Ay123 – Fall 2007

STELLAR STRUCTURE AND EVOLUTION

Problem Set 6

Due Wednesday, November 21, 2007

1. Consider a first order ordinary differential equation $dy/dx = f(x)$, where $f(x)$ is a known function, which we wish to solve for $y(x)$. The equation cannot be integrated analytically and we need to find the solution by numerical integration. We are going to evaluate the error in predicting $y(x_i + h)$ from a known initial value $y(x_i)$.

a) Show that the error for a single step using the Euler scheme (replacing derivatives by differences) for integration of such a first order ordinary differential equation is proportional to h^2, where h is the size of the step and the interval goes from 0 to 1.0.

b) Show that using a mid-point technique (i.e. a second order Runge-Kutta integration) makes the error of a step proportional to h^3.

We are now going to apply these techniques to integrate the first order ordinary differential equation

$$\frac{dy}{dx} = x^2 + sin(x)$$

over the interval in x from 0 to 1. (x is given in radians.)

c) Solve this equation analytically and plot $y(x)$ over that interval.

d) Next use a second-order Runge-Kutta numerical integration. Use 40 steps over the interval in x. Plot your result for $y(x)$. What is the largest error in y in absolute value and as a fraction of y ?

2. In class we established several properties of the Hayashi and the Henyey tracks for pre-main sequence stars. Consider a cool dense gas cloud containing material with the Solar chemical composition, so that $X = 0.90$. In particular we showed in class that freely collapsing fragments of gas clouds can first be considered "proto-stars" when they become fully ionized, which occurs when

$$\frac{R}{R_\odot} = \frac{45}{(1 - 0.2X)} \frac{M}{M_\odot}$$

Let us denote this point as the initial point, with time $t = 0$, with R_0 the radius at this point.

We showed that the gravitationally collapsing star with no nuclear energy sources follows the Hayashi track, which we take to be defined by $log(L) = 15\ log(T_{eff}) + 0.2\ log(M) + \delta$. The constant δ is to be set by a Solar mass star at $t = 0$ having $L/L_\odot = 40$ when its radius is R_0.

The pre-main sequence star switches to follow the Henyey track once it becomes hot enough to develop a radiative core. We assume that this switch is instantaneous.

We assume the Henyey track is defined by

$$\frac{L}{L_\odot} \propto [\frac{M}{M_\odot}]^{5.5}\ [\frac{R_\odot}{R}]^{0.5}$$

Since a pre-main sequence star has no nuclear energy generation, its energy is from gravitational contraction. We use the virial theorem to find L. Assume $L(t) = (1/2)\ d\Omega/dt$, and $\Omega = -\ 0.4\ G\ M^2/R$.

For purposes of this problem, we define the main sequence by the following:

$$L \propto M^3, \quad R \propto M^{(n-1)/(n+3)}$$

where n is the exponent of the energy generation rate, $\epsilon \propto \rho T^n$, $n = 4$ for the $p-p$ chain and 16 for the CNO cycle. We assume the switch to the CNO cycle happens instantaneously at $M = 2.0 M_\odot$.

We assume the Sun is on the main sequence.

Consider stars of mass 0.5 to 16 $M_\odot$ in steps of a factor of 2 in mass. Ignore both accretion after initial formation and mass loss.

a) (10 pts) The calculation starts by determining the HR diagram track, i.e. evolution in the plane $\log(L)$ versus $\log(T_{eff})$. For the specified values of stellar mass, first find the point in this plane where a pre-main sequence star of mass M must switch from the Hayashi track to the the Henyey track in order to reach the main sequence at its appropriate L and T_{eff}. Provide a table of the L, R and T_{eff} values at the switching point as a function of mass. Also plot an HR diagram with the evolutionary tracks for the pre-main sequence stars for each of the specified stellar masses, and with the main sequence, indicated.

b) (10 pts) Using the relations given above, find the time from $t = 0$ that the pre-main sequence star spends on the Hayashi track, on the Henyey track, and the total time to reach the main sequence as a function of mass over the specified mass range.

Comments on Techniques for numerical integration

The numerical solution of the differential equation,

$$dy/dx = f(x, y),$$

where $f(x, y)$ (the deriviative of the solution) is a known function of x and y, is discussed in many books on numerical analyses, including Numerical Recipes, chapter 15. There is also a brief discussion in your text by Hansen, Kawaler and Trimble in chapter 7 on stellar modelling.

To summarize, the Euler method uses the simple approximation that to integrate over an interval x_1 to x_2, one divides it into N steps of equal size h in x. If you know the value of $y(x_n)$, then to get the value of y at the next step $y(x_{n+1})$ you use the derivitive f calculated at the known initial point (x_n, y_n) to propagate the solution to the next step, i.e. $y_{n+1} = y_n + h \times f(x, y)$, with the last term evaluated at (x_n, y_n).

The second order Runge-Kutta technique uses instead a half step to the middle of the interval, and evaluates the function there, instead of at the beginning of the interval. Thus $k1 = h \times f(x, y)$ at (x_n, y_n). The the midpoint $(x, y[mid])$ is at $(x_n + h/2, y_n + k1/2)$. Now we calculate the value of y_{n+1} as $y_n + h \times f(x, y)$ at $(x, y[mid])$.

See the description in Numerical Recipes (section 15.1) for a more detailed discussion.