1. Radiative Transfer

We assume that you are familiar with the following, which we believe have already been covered in the course AY 121, radiative processes, which most of you are currently taking.

1.1. Radiative Transfer Definitions

the definition of the specific intensity $I(\nu, \bar{x}, \Omega, t)$, where Ω is a solid angle and $\hat{x}$ is the direction of the light beam, units: ergs/cm^2/hz/sec/unit solid angle.

$P(\text{rad}) = \int Icos^2(\theta)d\Omega/c$

energy flux $= \int Icos(\theta)d\Omega$

α (units: cm^{-1}, absorption coefficient/cm

mean free path $l = 1/\alpha$ (units: cm)

$\kappa_\nu =$ opacity/gm $= \alpha_\nu/\rho$ (units: cm^2/gm)

$\sigma_\nu =$ cross section/particle (units: cm^2)

$n\sigma_\nu \;=\; \alpha \;=\; \kappa_\nu\rho$ (n is the number density of the appropriate type of particle)

$j =$ the emission coefficient/cm.

$\eta =$ the emission coefficient/gm $= j/\rho$

1.2. Radiative Transfer in Stellar Atmospheres

Stellar atmosphere, M = total mass of star, R = total radius, g = surface gravity (constant) $= GM/R^2$. L is also fixed, we assume there are no sources of energy in the atmosphere. This is definitely true with regard to nuclear reactions.

We assume a plane parallel steady state static semi-infinite atmosphere with no incident radiation. We ignore sphericity effects (extended atmospheres).

z is the depth below the surface, $z = 0$ is the surface of the atmosphere, l is the length along an outgoing ray making an angle θ with respect to $\hat{z}$. r is the radial coordinate, which is 0 at the center of the star, not the surface. The equation of radiation transfer is:

$$\frac{\partial I}{\partial l} = cos(\theta)\frac{\partial I}{\partial z} = -\alpha I + j$$

Define the optical depth τ as $d\tau = \kappa\rho dz$, so τ, like z, increases inward from the surface and $\tau = 0$ at the surface. Note that τ is a dimensionless number.

We then transform the equation of radiative transfer:

$$\mu\frac{\partial I}{\partial \tau} = I - \frac{\eta}{\kappa} = I - S$$

where S is the source function, $S = \eta/\kappa$ and in this subfield, μ is the symbol conventionally used for $\cos(\theta)$. I, τ, κ, η and S are all functions of frequency ν.

In thermodynamic equilibrium, we know that $\partial I/\partial\tau = 0$, so $I = S = B_\nu(T)$, the black body Planck function.

The equation of radiative transfer has a formal solution, which for $\mu > 0$ is

$$I(\tau, \mu, \nu) = \int_\tau^\infty S_\nu(t) exp[-(t-\tau)/\mu]\frac{dt}{\mu}$$

At the surface, for $\mu < 0$, since there is no incoming flux, $I = 0$. For $\mu > 0$,

$$I(\tau = 0, \mu, \nu) \;=\; \int_0^\infty S_\nu(t) e^{-t/\mu} \frac{dt}{\mu}$$

The diffusion approximation, valid in stellar interiors, is basically an assumption that the gradient in T (temperature) is small, so that a series expansion for $S_\nu(t)$ can be limited to just the first two terms, $B_\nu(\tau) + (t-\tau)\frac{\partial B_\nu}{\partial \tau}$.

Integrating over depth the formal solution for $I(\tau, \mu, \nu)$ given above we get $I_\nu = B_\nu(\tau) + \mu \frac{\partial B_\nu}{\partial \tau}$.

Then integrating over angle to get the energy flux we obtain

$$L(r) \;=\; \frac{4\pi r^2}{3} \frac{\partial B}{\partial T} \frac{dT}{d\tau} \qquad L(r) \;=\; -\frac{16\pi a c r^2}{3\bar{\kappa}} \frac{dT}{dr}.$$

Although L is a function of frequency, the above equation is valid only for the integrated luminosity. We need to define a suitable κ averaged over frequency to use in the above equation. The Rosseland mean opacity is the appropriate one to use; it maintains the validity of the diffusion approximation at large depth. It is defined as:

$$\frac{1}{\tau_R} \;=\; \frac{\int_0^\infty (1/\kappa_\nu) \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}$$

1.3. Radiative Transfer Viewed as Momentum Transfer

Another derivation of the equation for $L(r)$ or for the energy flux $F(r) = L(r)/(4\pi r^2)$ can be derived by looking at the radiation pressure. The radiation pressure ($P(rad) = aT^4/3$) depends on the local temperature, while the flux, which is constant within the atmosphere, depends on T_{eff}. The force on an element due to the radiation pressure is given below. This force is equivalent to the energy absorbed/sec/unit length from the light beam. So we have

$$\frac{\partial P(rad)}{\partial r} = -\frac{4aT^3}{3}\frac{dT}{dr} = \frac{F\kappa_R\rho}{c}$$

This yields the same equation given above for $L(r)$ derived using a linear assumption for the dependence of $B_\nu(\tau)$ with τ.

1.4. Timescale for Photons to Escape From the Sun

The minimum opacity in the Solar interior is that from Thompson's scattering. Using the mean density of the Sun, assuming full ionization, we get $< n_e\sigma_e >= 0.5$ cm, yielding a mean free path l for photons of 2 cm. Assuming other sources of opacity increase σ by a factor of 10, so that the mean free path is then 0.2 cm.

The number of random walk steps required to get to the Solar surface from its center is $N = (R/l)^2$, and the time for a photon to escape is $Nl/c = R^2 l/c \approx 10^{12}$ sec, or 3×4 years.

1.5. Is Radiative Energy Transport OK ?

We consider whether radiative energy transport is sufficiently efficient to be able to carry the Solar energy flux from its center to the surface.

We ask if radiative energy transport can carry the known Solar luminosity, 2×10^{33} ergs/sec. We assume $dT/dr = T_c/R_\odot$ and evaluate the expression

$$L(rad) \;=\; 4\pi R_{\odot}^{2}[\frac{4\pi}{3\kappa_{R}\rho}acT^{3}\frac{dT}{dr}] \;\approx\; 10^{34} \text{ ergs/sec}$$

Since $L(rad) > L_{\odot}$, radiative energy transport can carry the entire Solar flux.

2. Approximations Useful for Stellar Atmospheres

We make the usual assumptions: semi-infinite, plane paralel layers, steady state, LTE...

2.1. The Eddington – Barbier Relations

We adopt the first two terms of a polynomial expansion for the source function S, $S = S_0 + S_1\tau$. This can be integrated, to give

$$I(\tau = 0, \mu, \nu) \;=\; S_0 + S_1\mu \;=\; S(\tau = \mu)$$

For normal incidence, I is given by S at $\tau = 1$ along the line of sight, and for slant paths, I is given by S at $\tau = \mu$.

The surface flux is then

$$\pi F_{\nu}(\tau = 0) \;=\; \int_{0}^{90} I(\tau = 0)\mu d\Omega \;=\; 2\pi\int_{0}^{1}(S_0 + S_1\mu)\mu d\mu \;=\; (S_0 + \frac{2}{3}S_1)\pi$$

$$F_{\nu}(\tau = 0) \;=\; (S_0 + \frac{2}{3}S_1) \;=\; S_{\nu}(\tau = 2/3) \;=\; B_{\nu}(\tau = 2/3)$$

With this approximation, the effective optical depth τ for the formation of the continuum light (i. e. I integrated over solid angle, which is the flux) is 2/3.

We have no information on the model structure $T(\tau)$, $\rho(\tau)$, etc. from these approximations; we have only derived properties of the radiation field.

The key results here are:

$$I(\tau = 0, \mu, \nu) \;=\; S(\tau = \mu)$$

$$F_\nu(\tau = 0) \;=\; B_\nu(\tau = 2/3)$$

2.2. Approximate Solution for $T(\tau)$

We go back to the concepts presented in §1.3. For the atmosphere, the flux is constant,

$$F \;=\; \sigma T_{eff}^4 \;=\; \frac{c}{\kappa_R \rho} \frac{\partial P(rad)}{\partial r}$$

Converting to a derivitive with respect to τ instead of r, we get:

$$\sigma T_{eff}^4 \;=\; c\frac{\partial P(rad)}{\partial \tau} \;=\; \frac{ac}{3}\frac{d(T^4)}{d\tau}$$

The constant a is $4\sigma/c$, so

$$T^4 \;=\; \frac{3}{4} T_{eff}^4 (\tau + q)$$

where q is a constant of integration.

We determine the constant q by noting that at the surface of the star there is no incident radiation, so $F(\tau = 0) = \sigma T_{eff}^4/2$. This determines q to be 2/3.

The key result is

$$T^4 \; = \; \frac{3}{4}T_{eff}^4(\tau + \frac{2}{3})$$

This determines the surface temperature, $T(\tau = 0)$ as $0.84T_{eff}$.

If we already have a fully converged detailed model atmosphere for a specific set of stellar parameters (T_{eff}, surface gravity, chemical composition) and wish to derive one for a similar star only slightly different in $Teff$, we can scale the reference model (denoted with superscript 0) to obtain an approximate solution. We expect

$$T(\tau) \; \approx \; \frac{T_{eff}}{T_{eff}^0} \, T^0(\tau).$$

The figure illustrates that this works reasonably well.

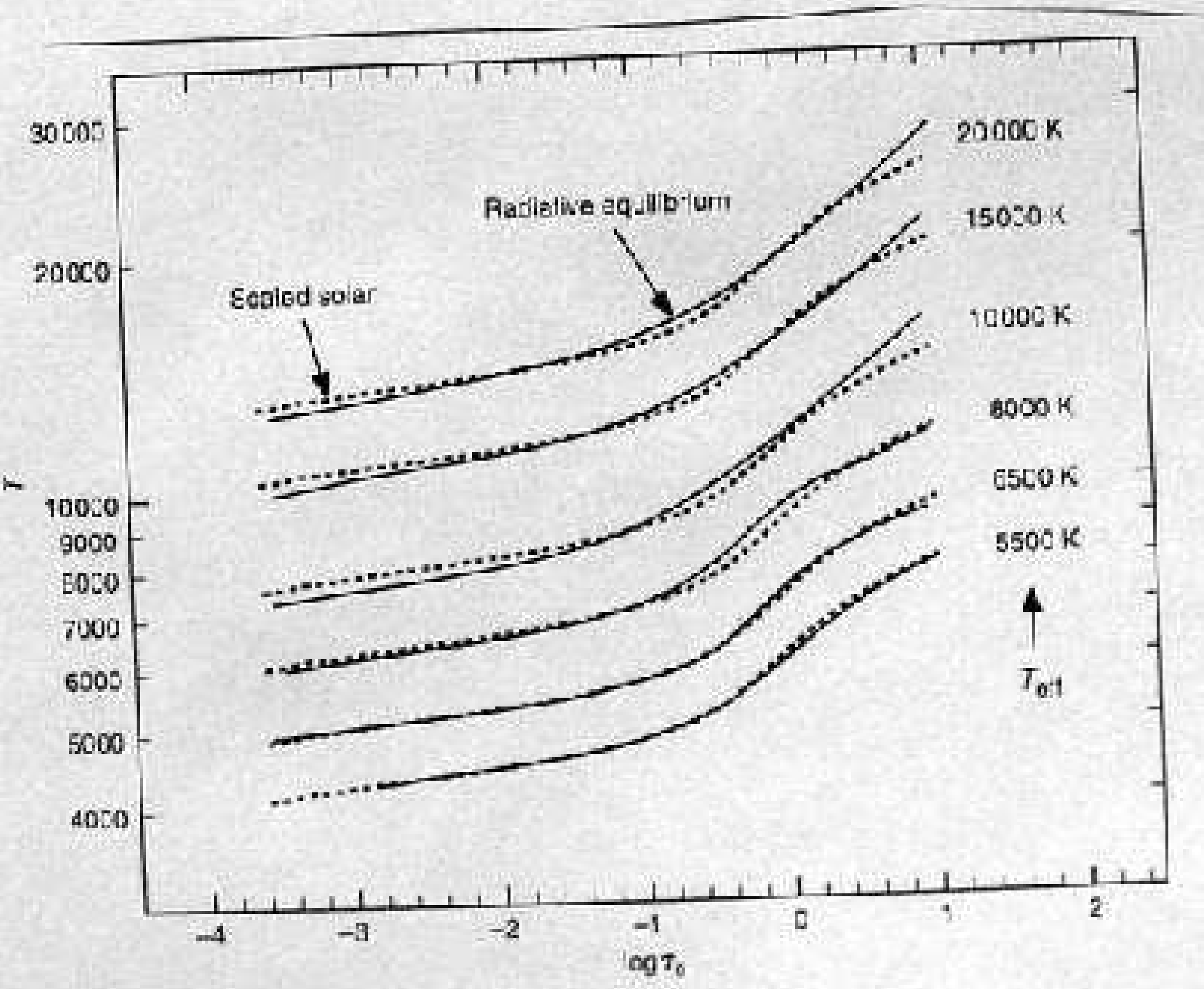

Fig. 9.5. Radiative-equilibrium models from the grid of Kurucz (1979) are compared to the scaled solar model from the same grid.

Fig. 1.— A comparison $T - \tau$ for scaled solar models with that derived from full detailed calculations of model atmospheres. (Fig. 9.5 of Gray)

2.3. Finding the Pressure

Once $T(\tau)$ is known, we use the equation of hydrostatic equilibrium to derive $\rho(\tau)$. Since we are using τ as the variable instead of r, we no longer need the minus sign in this equation.

$$\frac{dP}{d\tau} = \frac{g}{\kappa}$$

We guess a form for κ, then integrate for P. Next we calculate $P_e(\tau)$, then calculate $\kappa(\tau)$ from the solution for $T(\tau)$ and the initial solution for $P(\tau)$, solve again for P, repeat until convergence is achieved.

We define the column mass m (units: gm/cm^2) to be measured inward from the surface; $dm = \rho dz = -\rho dr$. The relationship between pressure and m is easily integrated, and we get

$$\frac{dP}{dm} = g \qquad P = gm + C$$

where C is a constant of integration, but we again need the opacity to convert between the column mass and the optical depth τ.

3. Energy Transfer by Various Processes

If multiple different types of radiative processes occur (i.e. free-free, bound free, Thomson scattering etc), the opacities are additive, and one simply sums up the relevant opacities for each process, so that $\alpha = \sum n_i \sigma_i(\nu)$.

However, if there are multiple modes of energy transport such as convection, conduction, etc. then what is summed is the flux transported by each so that $F(total) = L/(4\pi r^2) = F(rad) + F(conv) + F(cond) + F(?)$. Each of these has an effective κ, and in this case the total opacity must be found as:

$$\frac{1}{\kappa(total)} = \frac{1}{\kappa(rad)} + \frac{1}{\kappa(conv)} + \frac{1}{\kappa(cond)}$$

In other words, the flux takes the easiest way out of the star and the smallest "opacity" dominates the total κ for multiple mechanisms of energy transfer.