

1. A Gas

What is a gas ? Particles are free to move throughout the volume. Pressure is normal to any surface. Particle interactions are not important.

What is a fluid ? The total volume is fixed, but the fluid can flow and rearrange its shape. Shear forces can exist (viscosity) so the pressure force may not be perpendicular to an area selected within the volume.

What is a solid ? Particle interactions are very important. Particles can't move much. This leads to a fixed shape and a fixed volume.

Stars are (almost always) composed of gaseous material. Only very late in stellar evolution is this sometimes violated.

2. Motivation

Radioactive dating of the geologic fossil record demonstrates that life existed on Earth $\sim 4 \times 10^9$ yr ago. Liquid water is needed for our form of life, water freezes at 273 K and boils at 373 K, so the range of temperature on the Earth over that time span could not have been large, $|\langle \Delta T \rangle| \lesssim 5^\circ \text{ K}$.

This sets an important constraint on $\Delta(L_\odot)$ to be small, and the Sun must be STABLE over the required long timescale.

The total energy emitted by the Sun over a geologic time scale, E_{tot} , is

$$E_{tot} = L_\odot \Delta t = 3 \times 10^{33} \text{ ergs/sec} \times (3 \times 10^9 \text{ yr}) \times (3 \times 10^7 \text{ sec/yr}) = 3 \times 10^{50} \text{ erg}$$

If E_{tot} is converted to a mass, $E_{tot} = \Delta M c^2$, $\Delta M = 3 \times 10^{29}$ gm (assuming 100% efficiency converting mass to energy).

$\Delta M/M_{\odot} \sim 1.5 \times 10^{-4} \sim 0.02\%$ of the Solar mass. That is a small enough fraction that meeting the stability requirement should not be a problem.

3. Hydrostatic Equilibrium

For a spherically symmetric object, the gravity force at a given radius r points toward the center of the object, and the layers outside r do not count. Consider a small mass element m whose thickness is dr and whose area perpendicular to r is 1 cm^2 from within the star at r . The mass element m is then $(\rho dr \text{ 1 cm}^2)$. It experiences a gravity force $F_{grav} = GM_*(r)m/r^2$, where $M_*(r)$, which we sometimes abbreviate as $m(r)$, is the total mass of the star interior to the radius r .

The only resisting force (assuming no rotation or magnetic field) is a pressure gradient. A large pressure by itself is not sufficient. The pressure must increase as one moves inward within a star.

A large P gradient \rightarrow a large gradient in T and/or ρ .

So for static, stable, spherically symmetric stars, we require

$$\frac{dP}{dr} dr = -GM_*(r)m/r^2$$

in order for stability to hold. This is the equation of hydrostatic equilibrium.

This equation takes a special form near the surface of a star, as there $M_* = M_*(R)$ is the total mass of the star, and we get:

$$GM_*m/R^2 = -\frac{dP}{dr} dr, \quad \frac{dP}{dr} = -\rho g$$

where the latter holds only near the stellar surface, and g , the surface gravity, $= GM_*/R_*^2$.

The equation of hydrostatic equilibrium cannot be integrated, even near the surface, as we need an equation of state to specify the relationship between P and T . In some cases we can assume $P \propto \rho^m$, independent of T , where $m = 1 + 1/n$, and n is called the polytropic index. The adiabatic case, $PV^\gamma = \text{constant}$, is an example of a polytropic equation of state. When such an assumption is possible, the hydrostatic equilibrium equation combined with the equation of mass continuity can be solved (albeit with difficulty) for $P(r)$, $m(r)$, $\rho(r)$, etc. with no further information required. The solutions are messy functions, but they can be evaluated. Certain specific conditions are required for such polytropic models to be relevant to real stars. Such conditions are best met among stars with very high ρ , such as white dwarfs, neutron stars, etc.

The normal monotonic gas equation of state (the perfect gas law) is $P = nkT = \rho kT/(\mu m_H)$, where m_H is the mass of a hydrogen atom, n is the number of particles per unit volume, and μ is the mean atomic weight per particle in units of the weight of a H atom, defined as $\Sigma(n_p m_p)/\Sigma n_p$, where n_p is the number of particles of mass m_p (in units of m_H), and the sum is over all types of particles present in the gas, including for example, electrons, neutral atoms of all elements present in the gas, singly ionized atoms, doubly ionized atoms, etc. μ will be discussed later, values are: for pure neutral He, $\mu = 4$, while for pure neutral H, $\mu = 1$, and for pure fully ionized H, $\mu = 1/2$, since free electrons have a mass much less than m_H but still count as particles. A similar parameter, μ_e can be defined as the mean mass per free electron, defined similarly, so that $n_e = \rho/(\mu_e m_H)$.

We combine the eqtn of hydrostatic equilibrium and the perfect gas law to get:

$$\frac{dP}{dr} = -\frac{g\mu m_H P}{kT}$$

We still can't integrate this, as we don't know $T(r)$. So we need more equations or more assumptions.

Assume the matter is isothermal (T constant) (not a good approximation for a star, but not too bad for some other things, as we will see). We use h as the variable and consider plane parallel layers of material (at constant T). We get

$$\frac{dP}{Pdh} = \frac{d[\ln(P)]}{dh} = -\frac{g\mu m_H}{kT} = -1/H$$

So the pressure is exponentially decaying with height, $P \propto e^{-h/H}$, with a pressure scale height

$$H = \frac{kT}{\mu m_H g}.$$

We can't use this for a star, but the approximation T constant is not too bad for the atmosphere of the Earth. Putting in all the constants, setting $T = 273$ K, we find the pressure scale height $H = 8.7 \times 10^5$ cm = 8.7 km.

Comparison to the run of P, T, H as a function of height in the Earth's atmosphere (see data in Allen, among other places): for $h \lesssim 50$ km, H is within 25% of 8.7 km ! High in the Earth's atmosphere, the assumption that T is constant is not valid, and H increases rapidly.

Allen's Astrophysical Quantities, 4th edition.

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Table 11.21. Altitude profiles of mean physical conditions at latitude 45° [1].

Altitude (km)	log P (Pa)	T (K)	log ρ (kg m ⁻³)	log N (m ⁻³)	H ^a (km)	log l ^b (m)
0	+5.006	288	+0.0881	25.41	8.4	-7.2
1	+4.95	282	+0.0460	25.36	8.3	-7.1
2	+4.90	275	+0.00286	25.32	8.1	-7.1
3	+4.85	269	-0.0413	25.28	7.9	-7.0
4	+4.79	262	-0.087	25.23	7.7	-7.0
5	+4.73	256	-0.133	25.19	7.5	-7.0
6	+4.67	249	-0.180	25.14	7.3	-6.9
8	+4.55	236	-0.279	25.04	6.9	-6.8
10	+4.42	223	-0.384	24.93	6.6	-6.7
15	+4.08	217	-0.71	24.61	6.4	-6.4
20	+3.74	217	-1.05	24.27	6.4	-6.0
30	+3.08	227	-1.73	23.58	6.7	-5.4
40	+2.46	250	-2.40	22.92	7.4	-4.7
50	+1.90	271	-2.99	22.33	8.0	-4.1
60	+1.34	247	-3.51	21.81	7.4	-3.6
70	+0.72	220	-4.08	21.24	6.6	-3.0
80	+0.022	199	-4.73	20.58	6.0	-2.4
90	-0.74	187	-5.47	19.85	5.6	-1.6
100	-1.49	195	-6.25	19.08	6.0	-0.85
110	-2.15	240	-7.01	18.33	7.7	-0.10
120	-2.60	360	-7.65	17.71	12.1	+0.52
150	-3.34	634	-8.68	16.71	23.	+1.52
220	-4.07	855	-9.59	15.86	36.	+2.38
250	-4.61	941	-10.22	15.28	45.	+2.95
300	-5.06	976	-10.72	14.81	51.	+3.41
400	-5.84	996	-11.55	14.02	60.	+3.80
500	-6.52	999	-12.28	13.34	69.	+4.89
700	-7.50	1000	-13.51	12.36	131.	+5.86
1000	-8.12	1000	-14.45	11.74	288.	+6.49

Notes
^aH = pressure scale height (km).
^bl = mean free path (m).

Reference
 1. COESA, *U.S. Standard Atmosphere 1976*, (Government Printing Office, Washington DC)

Fig. 1.— Table 11.2 of Allen, *Astrophysical Quantities*, 4th edition, which gives the characteristics of the Earth's atmosphere as a function of height above the surface of the Earth.

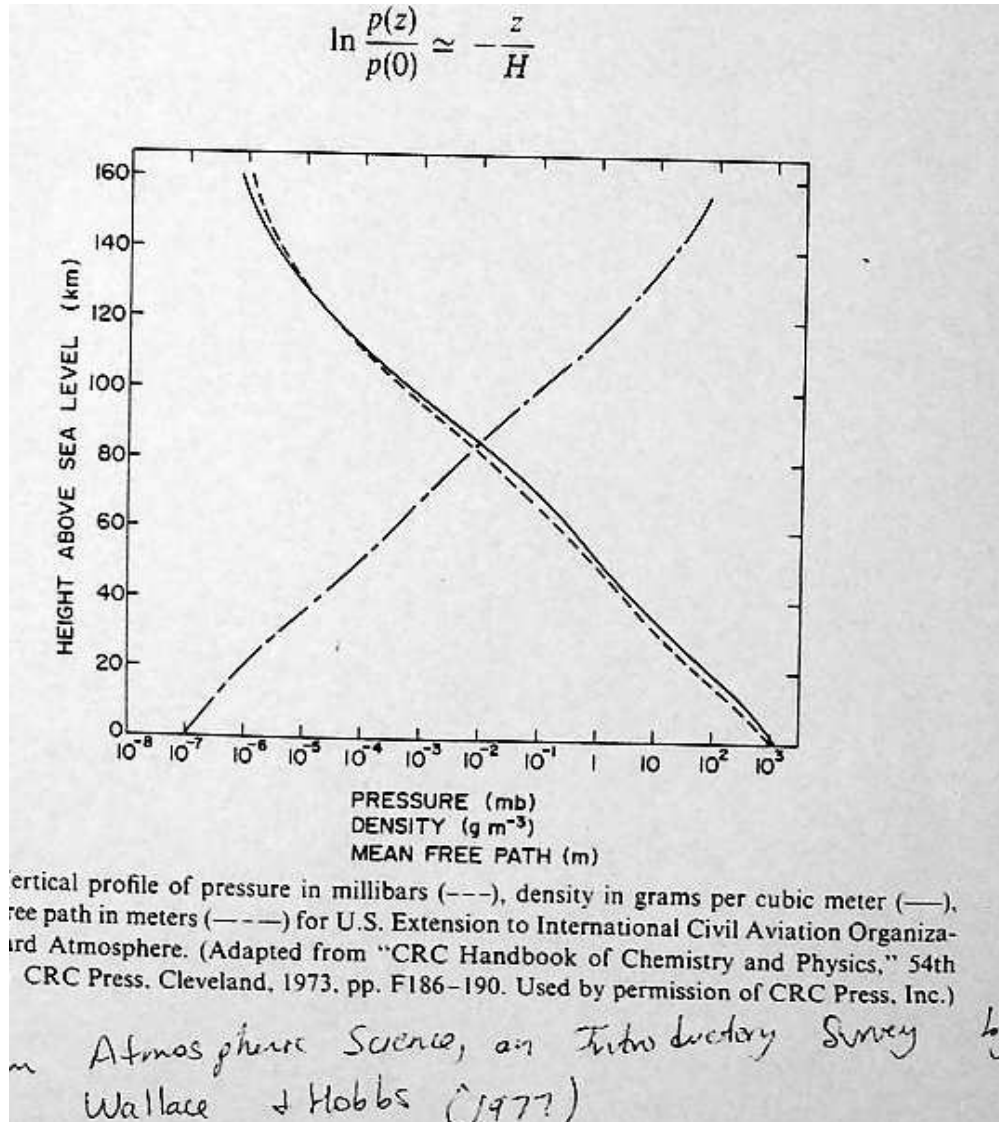


Fig. 2.— Vertical profile of pressure in millibars, density per cubic meter, and mean free path in meters for standard Earth atmosphere.

4. Mass Continuity

Another obvious equation can be derived by considering the distribution of mass in a spherically symmetric star. The variable r ranges from 0 to R , and the enclosed mass $M_*(r)$ (which we call $m(r)$) ranges from 0 to M_* , the total mass of the star.

Since the mass inside a thin spherical shell of thickness dr at radius r is $4\pi r^2 \rho(r) dr$, we then have

$$\frac{dm}{dr} = 4\pi r^2 \rho.$$

This is called the equation of mass continuity.

Substituting this into the hydrostatic equilibrium equation, we get

$$\frac{dP}{dr} = -\frac{Gm(r)}{4\pi r^4}$$

Gravitational Potential Energy

It is possible to derive the equation of hydrostatic equilibrium by considering the minimum of the total energy of a star to be its stable state, as is done in HKT. The total energy of a stable static star is the sum of the internal energy (i.e. the thermal energy) and the potential energy (the gravitational potential energy, Ω), equivalent to the work required to disassemble the star by moving shells of mass dm from r to ∞ . For such a shell, $d\Omega = -\frac{GM(r)}{r} dM(r)$, since work = \int_r^∞ Force $\times dl$ (where dl is distance along the radius) and the force at a radial distance l is $(GM(r)/l^2)dM(r)$. Dispersing the whole star, shell by shell, is equivalent to integrating over $dM(r)$, and we get

$$\Omega = \int_0^{M_*} d\Omega = -\int_0^{M_*} \frac{GM(r)}{r} dM(r) = -q \frac{GM_*^2}{R}$$

q is a form factor, related to the distribution of ρ with radius. It is a constant whose value is near 1. For constant ρ , $q = 3/5$.

$\Omega \approx -GM_*^2/R$. For the Sun, this is $\approx -3.8 \times 10^{48}$ ergs.

If the Sun were to radiate with its gravitational potential energy as the source of emitted energy, it would shine for a time $t_{KH} = -\Omega_{\odot}/L_{\odot} = 3 \times 10^7$ yr. This timescale, called the Kelvin-Helmholtz timescale, is very small compared to the geological fossil record and indicates that the source of energy for the Sun cannot be gravitational potential energy.

Total Energy The total energy of a stable static star $W = \Omega + \int_0^{M_*} E dM(r)$, where E is the internal energy/gm and U is the total internal energy for the star following the nomenclature of HKT. For an ideal monotonic (not molecular) gas, $E/\text{particle} = (3/2)kT$. The first law of thermodynamics, $PdV + dU = dQ$, where V is the volume of a parcel of gas, is a statement of conservation of energy. One then considers an adiabatic change, where “adiabatic” means constant entropy, no net heat flow, $dQ = 0$, so $PdV = -dU$. Another derivation of the eqtn of hydrostatic equilibrium proceeds from there (see KWT).

Rotating stars are a special case. We can assume that they are stable (i.e. the rotation rate changes very slowly compared to other timescales of interest), but not static. The simplest approach is to assume rotation on cylinders with angular velocity ω . Then an extra term appears in the total energy, and the surfaces of constant P no longer have spherical symmetry, and the star will not have a spherical surface.

Adiabatic Changes Suppose one considers a process which occurs on timescales short compared to that required to alter the temperature distribution within a star. In such a case we may consider that there is no perturbation to the normal heat flow within the star and $dQ = 0$ holds to a high level of accuracy. Such a process is called an adiabatic process, and has $PdV + dU = 0$.

An adiabatic gas has $PV^\gamma = \text{constant}$, where γ is called the adiabatic exponent, and $\gamma = c_P/c_V$, the ratio of the specific heat at constant pressure to that at constant volume. Such a gas has an equation of state $P = (\gamma - 1)\rho E$. A star composed of a gas which behaves adiabatically, i.e. has $PV^\gamma = \text{constant}$ throughout the star, has a total energy $W = \Omega(3\gamma - 4)/[3(\gamma - 1)]$.

Typical values of γ are 5/3 for a perfect monatomic gas, 4/3 for a relativistic gas, and 4/3 for radiation.

Dynamical Timescale

Dynamical timescale: How long does it take to restore hydrostatic equilibrium, given a perturbation ?

Recall that for a fluid in motion, $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v \cdot \Delta f$, where the first term is a partial derivative representing the change with time of the fluid at a fixed point, and the second term takes into account the motion of the parcel of fluid (gas). The radial force/acceleration equation is then:

$$\rho \left[\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} \right] = - \frac{dP}{dr} - \rho Gm(r)/r^2$$

If the right hand side is 0, the star is in hydrostatic equilibrium, and there is no radial motion. If not, approximate $\frac{dv_r}{dt} = -\epsilon Gm(r)/r^2$, where ϵ is a dimensionless imbalance parameter. $v_r \sim R_*/t_D$, where t_D , the dynamical timescale, is $t_D = \sqrt{R_*^3/(GM_*\epsilon)}$.

For the Sun, $t_D = 1.6 \times 10^3/\sqrt{\epsilon}$ sec, which is less than one hour for $\epsilon = 1$.

Thus even for a very small imbalance, $\epsilon \sim 10^{-8}$, $t_D = 1.6 \times 10^7$ sec (≈ 1.2 year).

Since the Sun is stable on a timescale of 10^9 yr = 10^{16} sec, $\epsilon < 10^{-26}$ for the Sun. (This is not true for pulsating variable stars.) If we set $\epsilon = 1$, then $t_D = 2 \times 10^3 (M/M_\odot)^{-1/2} (R/R_\odot)^{3/2}$ sec.

Free Fall Timescale

Suppose gravity is suddenly turned off or $P \rightarrow 0$ suddenly. Then $F \sim mr/t^2 = GM_*m/R_*^2$, where F is the radial force and where the acceleration has been replaced by r/t^2 . The resulting timescale is $t = \sqrt{R^3/(GM_*)} = 1.6 \times 10^3$ sec for the Sun.

Einstein Timescale

The Sun shines by radiating nuclear energy. Let f be the efficiency of nuclear reactions in converting mass into energy. Then $t_{nuc} = fM_*c^2/L_* = f \times 1.4 \times 10^{13}$ yr. We shall see that $f \sim 1\%$, so this gives a very long timescale, consistent with the geological record.

Stability Under Pressure Perturbations

For a small δP , star will react and adjust on a hydrodynamic timescale, much shorter than the timescale for $T(r)$ to change. So this is adiabatic, and we can consider $T(r)$ fixed.

If we end up with a 2nd order equation, $\frac{d^2(\delta P)}{dt^2} = -A\delta P$, and $A > 0$, the solution is oscillatory. If $A < 0$, the solution is an increasing exponential, and the star is unstable.

For stability, we need $A > 0$. The frequency of oscillation is $\omega = \sqrt{A}$, ω is then a real positive number.

Imagine a shell inside a star. Then from hydrostatic equilibrium,

$$P(r) = \int_{M(r)}^{M_*} \frac{GM(r)}{4\pi r^4} dM.$$

Now we have an adiabatic homologous compression. The right side of the above $\propto [R(\text{new})/R_0]^{-4}$. The left side, since $PV^\gamma = \text{constant}$, $P \propto V^{-\gamma} \propto R^{-3\gamma}$, and $\rho \propto R^{-3}$, so left side $\propto [\rho_{\text{new}}/\rho_0]^\gamma \propto [R_{\text{new}}/R_0]^{-3\gamma}$.

If $P_{\text{new}} > \text{rhs}$, the increase in P due to contraction of the star is bigger than the increase in the gravitational force. The star will then expand back out, and is stable.

This requires $P_{\text{new}}/P_0 > [R(\text{new})/R_*]^{-4}$, or $[R_{\text{new}}/R_0]^{-3\gamma} > [R(\text{new})/R_*]^{-4}$, and we have $R_{\text{new}} < R_0$.

The condition for stability is thus

$$\gamma > 4/3.$$

For an ideal gas, $\gamma = 5/3$, for pure radiation,, $\gamma = 4/3$. In an ionization zone, $\gamma \rightarrow 1$. We therefore anticipate stability problems in ionization zones.

Pulsational Timescale

Consider the perturbations in structure of a spherically symmetric star from a sound wave. This timescale is short compared to the timescale for changes in T , and hence this is an adiabatic process (to a high degree of approximation). If we assume the speed of sound v_s is constant within the star, a sound wave will travel from the center to the surface of the star and back over a time $\Pi = 2R/v_s$, where Π is the period of the pulsation, if such occurs.

$$v_s^2 = \frac{dP}{d\rho} \Big|_{ad} = \Gamma P/\rho, \quad \text{where } \Gamma = \frac{d \ln(P)}{d \ln(\rho)} \Big|_{ad} = (\rho/P) \frac{dP}{d\rho} \Big|_{ad}.$$

Γ is one of the adiabatic exponents, that known as Γ_1 . It is dimensionless and ~ 1 . We thus have $v_s \propto \sqrt{T}$.

Using the virial theorem, assumed to be approximately true throughout the minor perturbation of any pulsations, we get $-\Omega \approx (3v_s^2/\Gamma)M_*$. We then eliminate M_* and replace it with the mean density, and ignore constants of order unity, to derive

$$\Pi \approx \frac{1}{[G \langle \rho \rangle]^{1/2}} \approx \frac{0.04}{[\langle \rho \rangle / \langle \rho_\odot \rangle]^{1/2}} \text{ days.}$$

This is the same as the dynamical time derived above.

4.1. The Virial Theorem

Let us assume spherical symmetry, a static stable star with $P = 0$ at large R . For a monotonic ideal gas $U = \int_0^R 1.5kTN(r)dV = \int_0^R (1.5kT)/(\mu m_H)dM(r) = \int_0^R (1.5kT)/(\mu m_H)4\pi r^2 \rho dr$

Recall that the gravitational potential energy $\Omega = -\int_0^R (GM(r)/r)dM = -\int_0^R GM(r)4\pi r \rho dr$.

Now we use the equation of hydrostatic equilibrium, only including the gravitational and pressure gradient forces, so $\frac{dP}{dr} = -\rho GM(r)/r^2$. Multiply this by $4\pi r^3$ and integrate over r . We get $\int \frac{dP}{dr} 4\pi r^3 dr = -\int [\rho GM(r)/r^2] 4\pi r^3 dr$ The RHS is Ω , the gravitational potential energy. We integrate the LHS by parts, $\int \frac{dP}{dr} 4\pi r^3 dr = P(4\pi r^3) \Big|_0^R - \int_0^R 3P 4\pi r^2 dr$. The first term evaluates to 0, as we have assumed no external pressure. The second term is $-\int_0^R 3NktdV = -2U$.

So for a perfect gas in a spherical static stable star, we have

$$2U + \Omega = 0, \quad 2U = -\Omega \quad W = U + \Omega = \Omega/2 < 0.$$

The total energy W is negative as is expected for a bound system. If W were positive, the system would disperse.

A more careful derivation for a steady state star including magnetic and rotational energy results in the virial theorem $2E(\text{rot}) + 3(\gamma - 1)U + \Omega + E(\text{mag}) = 0$, where $E(\text{mag}) = \int B^2/(8\pi)dVol$ and $E(\text{rot}) \approx I\omega^2$, where I is the moment of inertia.

Consequences of the Virial Theorem

If star contracts, Ω becomes more negative, so U must become more positive, $\Delta U = -\Delta\Omega/2$. Then $\Delta W = \Delta U + \Delta\Omega = -|\Delta\Omega|/2$.

So half of the contraction energy goes into heating up the interior of the star (increasing U), and half is lost through radiation. The other half cannot go into increasing U , otherwise T would become too large, hydrostatic equilibrium could not be maintained, and the star would expand outward.

The star has a **negative** specific heat. It get hotter when it contracts, its total energy decreases when it contracts, and it is radiating away part of the contraction energy. This is also true of gas in clusters of galaxies, and is valid as long as viscosity can be ignored.

A perfect gas has a positive specific heat. It expands when it gets hotter. In a perfect gas, $|\Omega| \ll U$.

4.2. Guesstimates of Stellar Characteristics

Temperature

The mass averaged T of the Sun $\langle T \rangle$ is given by:

$$\langle T \rangle = \frac{\int_0^R T dm}{M_*} = \frac{\int_0^R T 4\pi r^2 \rho dr}{M_*},$$

so $M_* \langle T \rangle = \int_0^R 4\pi r^2 P \mu m_H dr / k = (\mu m_H / k) \int P dV$.

From the virial theorem, $2U = -\Omega$, and $|\Omega| \approx GM_*^2/R^2$. Substituting this into the above, we end up with

$$\langle T \rangle = \mu m_H GM_* / (6kR_*) = 1.8 \times 10^6 \text{ K for the Sun.}$$

The central T : We use the equation of hydrostatic equilibrium, $\frac{dP}{dr} = -\rho Gm(r)/r^2$, but substitute differentials between the center and surface for the derivative, assuming the surface pressure is negligible, and evaluate the right side very crudely, to get $\frac{P_c}{R_*} \approx \rho_c GM_* / R_*^2$, then use the perfect gas law to replace P_c by $\rho_c k T_c / (\mu m_H)$, and end up with $T_c \approx \mu m_H GM_* / R_* = 2 \times 10^7 \text{ K for the Sun.}$

The virial theorem gives another estimate of the mean T : $U = -\Omega/2 \approx GM_*/(2R_*^2) = (3/2)Nk \langle T \rangle$. But $N = M_*/(\mu m_H)$, so $\langle T \rangle = (GM_*)/(3R_*)(m_H / \mu) / k \approx 4 \times 10^6 (M_*/M_\odot) (R_*/R_\odot) (\langle \mu \rangle / 0.5)$, where $\langle \mu \rangle \sim 0.5$ for the Sun.

Pressure

Estimate of central pressure:

Theorem: $P + \frac{Gm(r)^2}{8\pi r^4}$ decreases with r .

Proof: use eq. hydrostatic equilibrium equation to show that d/dr of the above expression is $-Gm^2/(2\pi r^5)$ which is always < 0 .

Note that $Gm(r)/(8\pi r^4) \propto \rho_c r^3/r^4 \propto r^2$, so $Gm(r)/(8\pi r^4) \rightarrow 0$ as $r \rightarrow 0$. So $P_c > P_s + (GM_*^2/(8\pi R_*^4))$ and at the surface $P_s \rightarrow 0$. Then $P_c > GM_*/(6R_*) < \rho >$.

Substituting in the mean density of the Sun, about 1.41 gm/cm³ (we know its mass and radius, hence $< \rho >$), and the other solar parameters, we get $P_c > 4.5 \times 10^{14}(M/M_\odot)^2(R/R_\odot)^{-4}$ dynes/cm², or $4.5 \times 10^8(M/M_\odot)^2(R/R_\odot)^{-4}$ atmospheres.

Another guess at P_c : replace the equation of hydrostatic equilibrium with one-zone differences, and set $P_s = 0$, so $\frac{P_c}{R_*} \approx 2 < \rho > GM_*/R_*^2$, so $P_c = 2GM_*/R_* = 5 \times 10^{15}$ dynes/cm² for the Sun. (Numerical models for the Sun give $P_c \sim 2 \times 10^{17}$ dynes/cm².)

5. Energy Generation and Transport

We here make a very simple first pass at energy generation and energy transport from the center to the surface of stars. We assume energy is generated only from nuclear reactions, which only occur close to the center of the star, where T reaches its highest value. Let ϵ be the energy generation rate/gm (units ergs/sec/gm). Then the total power generated within a spherical shell, assuming $\epsilon \approx$ constant, is $4\pi r^2 \rho \epsilon dr = \epsilon dm(r)$. This energy generated must be balanced by the loss from radiation from the surface, the luminosity of the star, otherwise the star is not static. (Over long timescales the star is *cannot* be static, as nuclear reactions change the mean particle weight and nuclear fuels become depleted, but that is over the very long Einstein timescale.) Then $\frac{dL}{dr} = 4\pi r^2 \rho \epsilon$. This is the energy generation equation. Of course, ϵ is a complicated function of T, P, X, Y, Z , etc. which we will study later. To make progress, we assume a parametric form: $\epsilon = \epsilon_0 \rho^\gamma T^\nu$, where the powers γ and ν depend on which nuclear process dominates in a particular star or region of a star.

Key pairs of values of γ and ν for three common nuclear processes are: p–p chain, 1, 4; CNO-cycle 1, 15; triple- α 2, 40. The first two of these burn H into He, while the third burns three He nuclei into ^{12}C . Note the very high power for T , with sensitivity to density of ρ or ρ^2 , as we might expect. T^{40} for the triple- α process means that a 10% increase in T leads to a factor of 45 increase in the nuclear energy generation rate !

Ignoring gravitational contraction and other non-nuclear energy sources, $L(r)$ must increase outward, until the r beyond which the gas is too cool for nuclear reactions to occur is reached, beyond which $L(r) = L(R) = \text{constant}$.

We still need a description of how this energy is transported from the central region of the star, where it is produced, to the surface. We ignore convection and diffusion and concentrate on radiative transport in this rough initial effort. We assume a black body law describes (at least approximately) the radiation field, so that the radiative flux $\propto T^4$. Then the radiative flux obeys a diffusion law, $F(r) = -D \frac{dE(\text{Rad})}{dr} = -D \frac{d(aT^4)}{dr}$. The coefficient D is going to depend on the properties of the material, with the key parameter being how transparent the material is to radiation. The relevant parameter is κ , the opacity, and $D \propto 1/\kappa$.

But $L(r) = 4\pi r^2 F(r)$, so

$$L(r) = - \frac{(4\pi r^2)^2 c}{3\kappa} \frac{d(aT^4)}{dr}.$$

κ is one of those messy parameters which depends on ρ, T, X, Y, Z , but we again adopt a parametric form, $\kappa = \kappa_0 \rho^\eta T^{-s}$ (units cm^2/gm). For completely ionized material characteristic of the stellar interior, Thomson scattering dominates, and $\eta = s = 0$, while in cooler parts of the star, Kramer's opacity ($\eta = 1, s = 3.5$) is a reasonable approximation.

6. Stellar Dimensional Analysis

We now have, at least in preliminary form, the four basic equations of stellar structure, hydrostatic eq., mass continuity, energy generation, and energy transport. We also have, at least in a preliminary power law form, the parameters that occur in these equations which characterize the gas, including the opacity and the energy generation rate. These form a closed set of equations which are in principle sufficient to solve for the stellar model, $P(r), \rho(r), T(r)$, etc.

However, the solution of this set of equations is difficult and cannot be expressed as a closed form of functions. In order to gain insight we look at the case of small variations around a previously known solution, i.e. homologous variations, where there are no abrupt discontinuities or changes, just gradual evolution of the equilibrium solution as one or more parameters are varied.

The specific assumption made to construct a set of homologous models is that any model is related to an initial one by a simple change in scale, so that the radial variable r is $(R/R_0)r_0$, where R is the radius of the new stellar model and R_0 is the radius of the initial one, $m(r) = (M_*/M_{*,0})m(r, 0)$.

We parameterize the relevant variables in power law form, $P = P_0\rho^{\chi_\rho}T^{\chi_T}$, and recall that (hydrostatic eq.) $P \propto M^2/R^4$.

The goal is to construct relations between R , ρ , T and L as a function of total stellar mass M of the form $L \propto M^{\alpha_L}$, where we need to solve for the exponents α_L, α_R , etc as functions of χ_ρ, χ_T (related to the powers of ρ and T in the equation of state), and the parameters η and s (describing κ) and λ and ν , describing the behavior of the energy generation rate, whose typical values were given above.

Note that α_L specifies the mass – luminosity relation for the set of stars homologous

to our initial model, an important theoretically predicted relation which can be directly compared to observations of real stars.

We find, for example, assuming radiative energy transport, that $\alpha_R = 0.333[1 - 2(\chi_T + \nu - s - 4)]/D_{rad}$, where D_{rad} has the powers $(3\chi_\rho - 4)((\nu - s - 4) - \chi_t)(3\lambda + 3n + 4)$. If we use Thomson scattering for the opacity and energy generation via the CNO cycle, both characteristic of massive stars, we find (for hydrogen burning upper main sequence stars)

$$R/R_\odot \propto (M/M_\odot)^{0.75}, \quad L/L_\odot \propto (M/M_\odot)^{3.5}.$$

Figures below show the $L - M$ and $R - M$ relationship for main sequence stars, and we see that the predicted relations derived above are a good fit.

For the lower main sequence stars, those less massive than the Sun, we need the $p - p$ H burning chain and Kramer's opacity. We also probably should consider convective energy transfer, but we'll ignore it. We then find $\alpha_L \sim 5.5$, which is a good fit to stars with masses near that of the Sun.

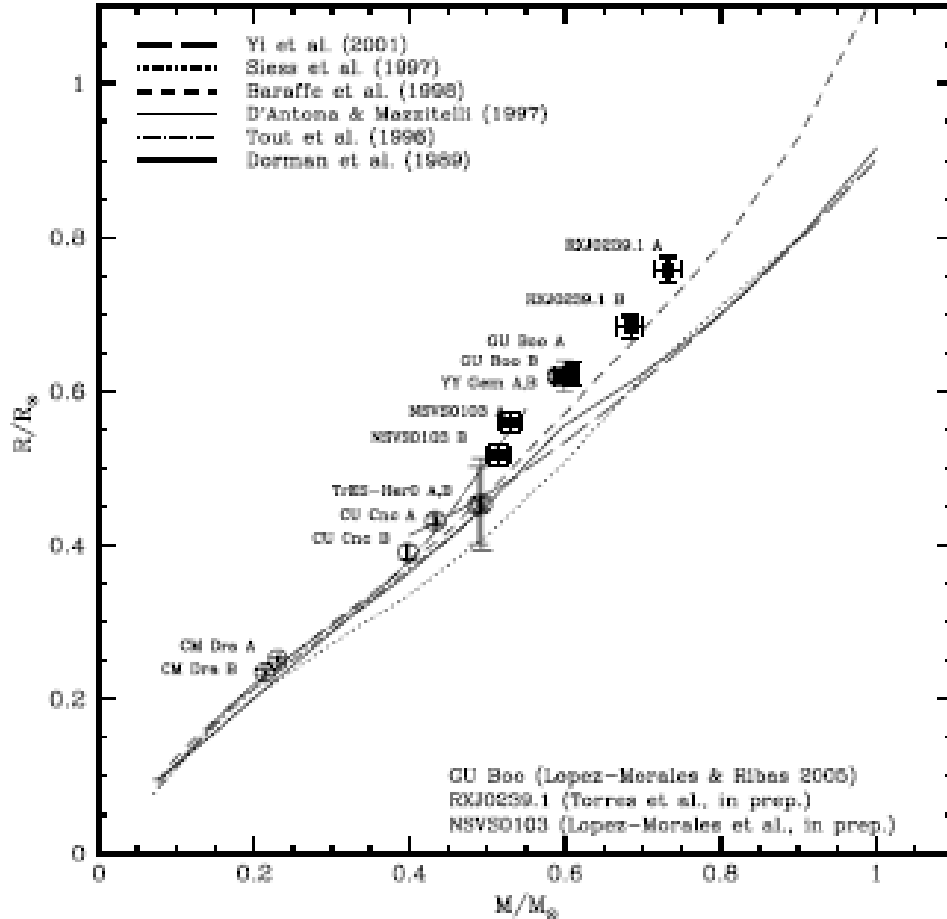


Figure 1. Mass-Radius relation of stars below $1M_{\odot}$. The lines represent different models for an age of 0.35 Gyrs and a metallicity of $Z = 0.02$. The open circles show the location of the binaries CM Dra, CU Cnc, TrES-Her0-07621, and YY Gem. The filled squares correspond to GU Boo, RX0239.1, and NSVS01031772. (NOTE: The mixing length used in the Baraffe model is $\alpha = 1.0$).

Fig. 3.— Mass-radius relation for stars with $M < M_{\odot}$ based on eclipsing binary systems.

Fig.1 of Lopez-Morales, Astro-ph/0603748.

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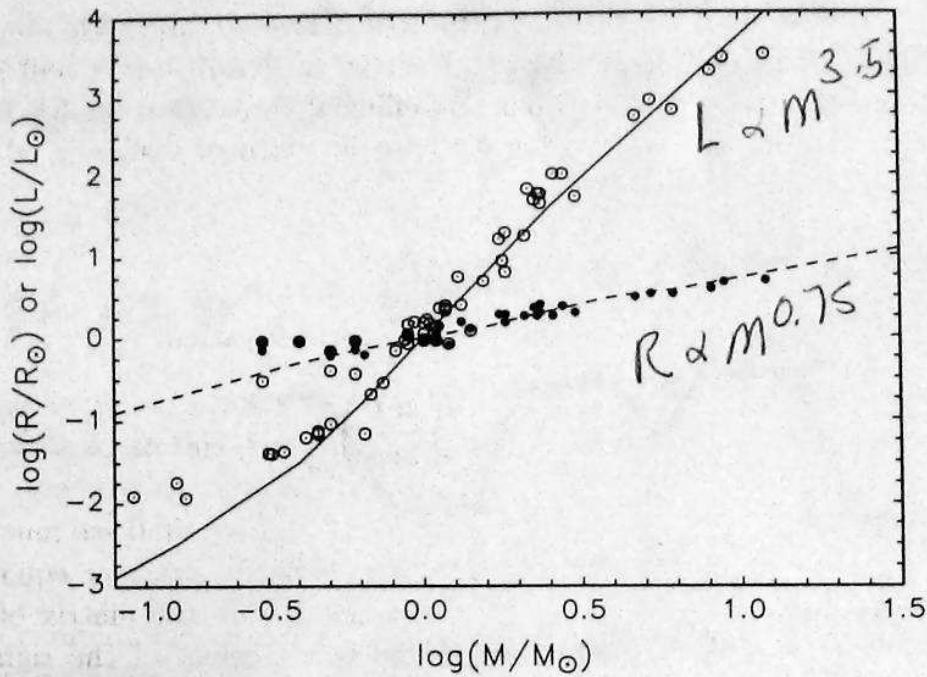


Fig. 1.3. Luminosity and radius versus mass for main sequence stars. All quantities are in solar units. The solid (dashed) line is that for luminosity (radius) adapted from material in Allen (1973). Open (filled) dots are luminosity (radius) for components of binaries, from Harris et al. (1963) and Böhm (1989). (from HK)

Fig. 4.— Mass- L and R relations for stars over a wide range in mass. The approximate exponent of a power law fit for stars with $M > 1M_{\odot}$ is indicated. This is Fig. 1.3 of HK.

6.1. Main Sequence Lifetimes

Having solved the homologous case as an approximation to the full set of stellar structure equations, we can use the results to derive main sequence lifetimes. We assume that the main sequence phase ends when 10% of the initial H in a star has been converted to He. (Clearly this number must be less than 100% as only the core of a main sequence star gets hot enough for nuclear reactions.) We further assume that initially the star is 70% H by mass, and that a fraction f of the mass is converted into energy and radiated through nuclear reactions. Then the main sequence lifetime becomes

$t_{MS} = (0.70 \times 0.10 \times f \times Mc^2)/L$ sec. If we use $f = 1\%$, we get

$$t_{MS} \approx 10^{10}(M/M_{\odot})(L/L_{\odot})^{-1} \text{ years.}$$

We then substitute the $M - L$ relations derived above, to get, for example, $t_{MS} \approx 10^{10}(M/M_{\odot})^{-2.5}$ years for upper main sequence stars. Since these may have M up to $100 M_{\odot}$, the main sequence lifetime for very massive stars, t_{MS} can be as small as 10^5 yr, while the Solar t_{MS} is $\approx 10^{10}$ yr.

Because of the high power of the stellar mass in the $L - M$ relationship, the main sequence lifetime decreases sharply as M increases.