Solution: HW 1
AY 123, Fall 2007

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Problem 1:

1a.

The star has density profile $\rho(r) = \rho_c(1 - r/R)$. The total mass (M) must be

$$\begin{aligned} M &= m(R) = \int_0^R 4\pi R^2 \rho(r) dr && (1) \\ &= \int_0^R 4\pi R^2 \rho_c \left(1 - r/R\right) dr && (2) \\ &= 4\pi R^3 \rho_c \int_0^1 x^2 \left(1 - x\right) dx && (3) \\ &= 4\pi R^3 \rho_c \left(1/3 - 1/4\right) && (4) \\ &= \frac{\pi}{3} R^3 \rho_c && (5) \end{aligned}$$

Therefore,

$$\boxed{\rho_c = \frac{3M}{\pi R^3}} \qquad (6)$$

1b.

From the hydrostatic equilibrium equation, $\frac{dP}{dr} = \frac{-Gm(r)\rho(r)}{r^2}$, we can determine the pressure profile, $p(r)$. First we need the mass enclosed within radius r.

$$m(r) = 4\pi \int_0^r \rho(r) r^2 dr = 4\pi\rho_c \left(\frac{r^3}{3} - \frac{r^4}{4R}\right) \tag{7}$$

Thus, we get for the pressure (integrating from $r = r$ to $r = R$, where $p(R) = 0$),

$$\begin{aligned} p(r) &= \int_r^R \frac{4\pi G \rho_c^2}{r^2} \left(\frac{r^3}{3} - \frac{r^4}{4R}\right)\left(1 - \frac{r}{R}\right) dr & (8) \\ &= \frac{36GM^2}{\pi R^4} \int_{r/R}^1 \left(\frac{x}{3} - \frac{x^2}{4} - \frac{x^2}{3} + \frac{x^3}{4}\right) dx & (9) \\ &= \frac{36GM^2}{\pi R^4} \left[\frac{x^2}{6} - \frac{7x^3}{36} + \frac{x^4}{16}\right]_{x=r/R}^{x=1} & (10) \\ &= \frac{5GM^2}{4\pi R^4} \left[1 - \frac{4}{5}\left(6x^2 - 7x^3 + \frac{9x^4}{4}\right)\right] & (11) \end{aligned}$$

therefore, $\boxed{p(r) = p_c f(x)}$ where $p_c = \frac{5GM^2}{4\pi R^4} = 4.4 \times 10^{15} \frac{(M/M_\odot)^2}{(R/R_\odot)^4}$dyne cm^{-2} and $f(x) = 1 - \frac{4}{5}\left(6x^2 - 7x^3 + \frac{9x^4}{4}\right)$. Note that this value of p_c is much smaller than the Sun's central pressure, 2.4×10^{17}dyne cm^{-2}, since the Sun is much more centrally concentrated.

1c.

From the ideal gas EOS,

$$p = nkT = \frac{\rho}{m_p \mu} kT \tag{12}$$

where n is the particle number density, m_p is the proton mass, and $\mu \approx 0.6$ is the mean molecular weight (HK 1.52). Thus, evaluating the temperature at the center of the star, T_c,

$$\begin{aligned} T_c &= \frac{m_p \mu p_c}{k \rho_c} & (13) \\ &= \frac{m_p \mu 5GM^2 \pi R^3}{k 4\pi R^4 3M} & (14) \\ &= \frac{5 m_p \mu GM}{12kR} & (15) \end{aligned}$$

$$\boxed{T_c = 5 \times 10^6 \frac{(M/M_\odot)}{(R/R_\odot)} \text{K}} \tag{16}$$

which is about a third of the actual solar value.

1d.

The radiation pressure at the center is $p_\gamma = \frac{1}{3}aT_c^4$. Plugging in T_c from 1c yields $p_\gamma = 2.8 \times 10^{12} \frac{(M/M_\odot)^4}{(R/R_\odot)^4}$dyne cm^{-2}. Therefore,

$$\frac{p_\gamma}{p_{gas}} = 6.3 \times 10^{-4} \left(\frac{M}{M_\odot}\right)^2 \tag{17}$$

So the gas pressure dominates as long as $M < 40 M_\odot$. Heavier stars with this density profile will be radiation pressure dominated in their cores.

1e.

The gravitational binding energy is (HK 1.6)

$$\begin{aligned}
\Omega &= -\int_0^M \frac{Gm(r)}{r} dm && (18)\\
&= -\int_0^R \frac{Gm(r)}{r} 4\pi\rho_c \left(1 - \frac{r}{R}\right) r^2 dr && (19)\\
&= -\left(4\pi\rho_c\right)^2 G \int_0^R \left(\frac{r^3}{3} - \frac{r^4}{4R}\right)\left(1 - \frac{r}{R}\right) r dr && (20)\\
&= -\frac{144GM^2}{R} \int_0^1 \left(\frac{x^4}{3} - \frac{x^5}{3} - \frac{x^5}{4} + \frac{x^6}{4}\right) dx && (21)\\
&= \frac{-144GM^2}{R} \left[\frac{1}{15} - \frac{1}{18} - \frac{1}{24} + \frac{1}{28}\right] && (22)
\end{aligned}$$

$$\boxed{\Omega = -\frac{26GM^2}{35R}} \tag{23}$$

The stellar kinetic energy is (HK 1.20)

$$\begin{aligned}
K &= \frac{3}{2}\int \frac{p(r)}{\rho(r)} dm && (24)\\
&= \frac{3}{2}\int_0^R p(r) 4\pi r^2 dr && (25)\\
&= \frac{6GM^2}{R}\int_0^1 \left(\frac{5x^2}{4} - 6x^4 + 7x^5 - \frac{9x^6}{4}\right) dx && (26)
\end{aligned}$$

So, $K = \frac{13GM^2}{35R}$, as is expected from the virial theorem, $K = -\frac{1}{2}\Omega$. Note that this result is fully general and independent of the EOS.

Problem 2:

The radiant flux received from the Sun is the Solar luminosity divided by the surface area over which it is spread:

$$F = \frac{L_\odot}{4\pi a_\oplus^2} \tag{27}$$

where the Solar luminosity is calculated from

$$L_\odot = 4\pi R_\odot^2 \sigma T_{\text{eff}}^4 \tag{28}$$

Combining the two expressions gives

$$F = \sigma T_{\text{eff}}^4 \left(\frac{R_\odot}{a_\oplus}\right)^2 \tag{29}$$

We note that $R_\odot/a_\oplus$ is the angular radius of the Sun, so the above gives the effective temperature in terms of given quantities:

$$T_{\text{eff}} = \left(\frac{F}{\sigma\theta_\odot^2}\right)^{1/4} \tag{30}$$

Plugging in the given values reveals

$$\boxed{T_{\text{eff}} = 5800\,\text{K}} \tag{31}$$

Problem 3:

The total energy of a star of mass M and radius R with the density profile in problem 1 is $E_{tot} = \Omega + K = -1.41 \times 10^{48} \frac{(M/M_{\odot})^2}{(R/R_{\odot})}$erg. Now, if the star has a surface temperature T_{eff}, it radiates energy at a rate of $L = 4\pi R^2 \sigma T_{eff}^4 = 1.35 \times 10^{32} \left(\frac{R}{R_{\odot}}\right)^2$erg/s.

Now, suppose the star begins with a very large radius and then contracts at a constant mass while radiating energy. We then have,

$$L = -\frac{dE_{tot}}{dt} = \frac{E_{tot} dR}{R dt} \tag{32}$$

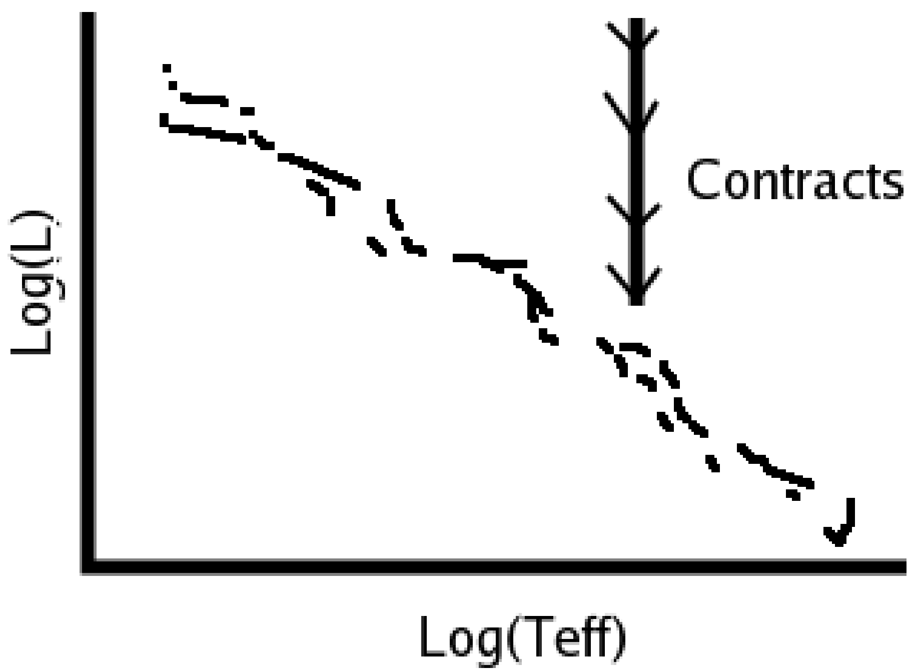
Plugging in, and letting $x = \frac{R}{R_{\odot}}$

$$dt = \frac{3.8 \times 10^{48} dx}{1.35 \times 10^{32} x^4} \tag{33}$$

$$t = 2.81 \times 10^{16} \int_1^{\infty} \frac{dx}{x^4} \text{ sec} \tag{34}$$

$$\boxed{t = 3.0 \times 10^8 \text{ years}} \tag{35}$$

Therefore, if the star contracts at constant T_{eff} with $L \propto T_{eff}^4 R^2$, then it begins very luminous and then becomes fainter as it contracts. We thus have the following HR diagram:


Contracts
Log(L)
Log(Teff)

Problem 4:

The period P is defined as $P = 2R/v_s$ where the sounds speed v_s is given by $v_s^2 = \Gamma_1 \frac{p}{\rho}$ and $\Gamma_1 = \left(\frac{d\ln p}{d\ln \rho}\right)_{ad}$.

From the virial theorem we know $\Omega \approx GM^2/R$, so

$$\Omega = 3\int \frac{p}{\rho}dm = 3\int \frac{v_s^2}{\Gamma_1}dm \tag{36}$$

$$\approx 3\frac{v_s^2}{\Gamma_1}M \tag{37}$$

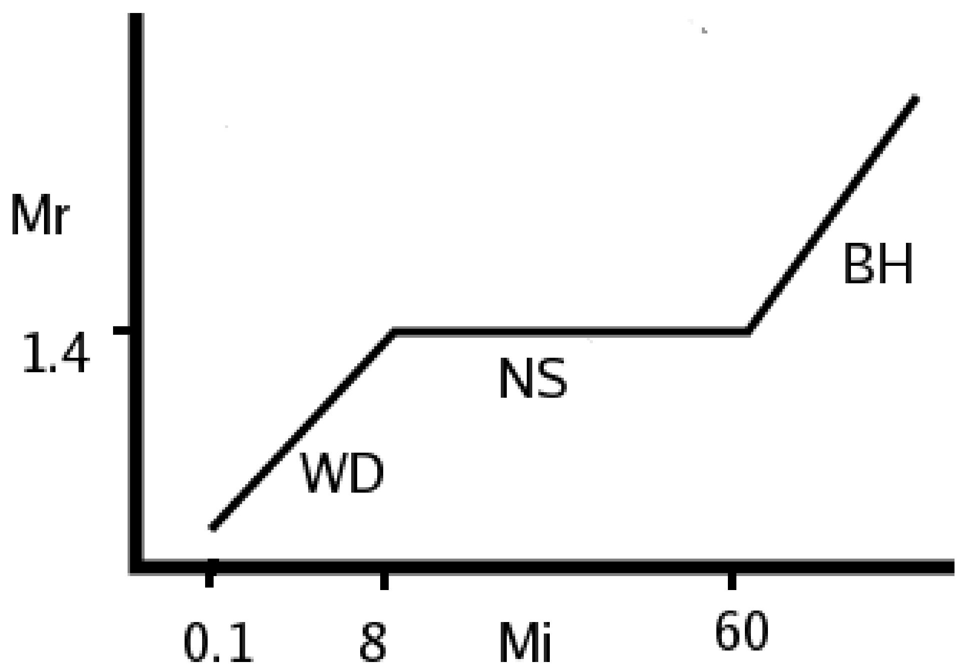
$$\text{so we know } 3\frac{v_s^2}{\Gamma_1}M \approx \frac{GM^2}{R} \tag{38}$$

Thus we plug in the expression for v_s and solve for P and find $P \sim \left(\frac{GM}{R^3}\right)^{-1/2} \sim (G\bar{\rho})^{-1/2} \sim \boxed{0.02\,(\bar{\rho}/\bar{\rho}_\odot)^{-1/2} \text{ days.}}$ It is useful to remember that this is about an hour for the sun.

Now, $P \propto \bar{\rho}^{-1/2} \propto \left(\frac{M}{R^3}\right)^2$ and using fixed T_{eff}, $L \propto R^2 T_{eff}^4$ and $L \propto M^3$, we see that $P \propto L^{7/12}$

Problem 5:

There are very detailed models for finding the remnant mass given an initial mass, but I will use rough numbers to keep it simple. The stellar remnant (M_r) versus initial mass (M_i) relation is such that $M_r \approx \left(\frac{M_i}{5}\right)$ for $0.1 \leq M_i \leq 8M_\odot$, $M_r \approx 1.4M_\odot$ for $8 \leq M_i \leq 60M_\odot$. For $M_i > 60M_\odot$, the relationship isnt as well known, but can be ignored because there is very little initial mass at these values for a Salpeter IMF.



Now, lets find the total initial stellar mass for the population given some proportionality constant a for the Salpeter IMF.

$$M_{tot,i} = \int M_i dN = \int_{0.1}^{60} M_i \frac{dN}{dM_i} dM_i \tag{39}$$

$$= a \int_{0.1}^{60} M_i M_i^{-2.35} dM_i \tag{40}$$

$$= \frac{a}{0.35} \times \left(0.1^{-0.35} - 60 - 0.35\right) \approx \boxed{5.71a} \tag{41}$$

After 10 Gyr, stars with masses larger than a solar mass will have evolved off the main sequence. The total mass left in the remaining stars and stellar remnants is

$$M_{tot,f} = a\int_{0.1}^{1} M_i M_i^{-2.35} dM_i + a\int_{1}^{8} \frac{M_i}{5} M_i^{-2.35} dM_i + a\int_{8}^{60} 1.4 M_i^{-2.35} dM_i \tag{42}$$

$$= 3.54a + 0.3a + 0.06a = 3.9a \tag{43}$$

Thus, $1 - \frac{M_{tot,f}}{M_{tot,i}} = 40\%$ is returned to the ism.

Problem 6:

This problem is easiest to calculate from the point-of-view of the binary system. Centre a celestial sphere on the binary system, oriented such that the system's angular momentum vector points North. The Solar system's location on the sphere is described by two angles θ and ϕ as in spherical coordinates. Note that the inclination angle i is equal to θ.

Assuming a random orientation, the probability of falling at a given θ and ϕ is $1/4\pi$. The probability of falling at a given i is

$$p(i) = \frac{1}{4\pi} 2\pi \sin i = \frac{1}{2} \sin i \tag{44}$$

The average value of $\sin^3 i$ is then

$$\left\langle \sin^3 i \right\rangle = \frac{1}{2} \int_0^\pi \sin^3 i \, \sin i \, \mathrm{d}i \tag{45}$$

which reveals

$$\boxed{\left\langle \sin^3 i \right\rangle = 0.589} \tag{46}$$

If the inclination is not known, then Kepler's 3rd Law for binary stars takes the form

$$(M_1 + M_2) \sin^3 i = \frac{(2\pi)^2}{G} \frac{a^3}{P^2} \tag{47}$$

Given luminosities measured through other means and a large enough sample, this allows one to find the average mass for a star of a given luminosity. The mass-luminosity relation follows.

Problem 7:

NOTE: In preparing the solution to this problem, I assumed the parallax had the same uncertainty as other angular measurements. It is okay if you assumed the measurement to be exact, as stated in the problem.

Recall: Uncertainties are added in quadrature. To find the error of some function $f(x_1, x_2, ..., x_n)$:

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \Delta x_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \Delta x_2^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \Delta x_n^2} \tag{48}$$

Which, for a function $f(x_1, x_2, ..., x_n) = const \times x_1^a \times x_1^b \times \cdots \times x_n^z$ reduces to

$$\frac{\Delta f}{f} = \sqrt{\left(\frac{a\Delta x_1}{x_1}\right)^2 + \left(\frac{b\Delta x_2}{x_2}\right)^2 + \cdots + \left(\frac{z\Delta x_n}{x_n}\right)^2} \tag{49}$$

7a.

From the parallax we know $d = 1\text{pc/arcsec}$, so $\boxed{d = 3 \times 10^{19}\text{cm}}$

The angluar size of the radius is $\theta = 2R_\odot/d = 4.7 \times 10^{-4}\text{arcsec}$

The angular semi-major axis is given by $\theta_a = 500R_\odot/d = 0.24\text{arcsec}$

Now we need to find the error in $r = \theta_r/\Pi$, where Π is the parallax, which is $\frac{\Delta r}{r} = \sqrt{\left(\frac{\Delta\theta_r}{\theta_r}\right)^2 + \left(\frac{\Delta\Pi}{\Pi}\right)^2}$. Using $\Delta\theta_r = 0.01$ and $\Delta\Pi = 0.01$ yeilds:

$$\boxed{\frac{\Delta R}{R} = 2130\% \text{ and } \frac{\Delta a}{a} = 10.9\%} \tag{50}$$

7b.

These curves are roughly the BB curves at various temperatures. Unfortunately, for $T = 5800\text{K}$ we are not safely in the RJ portion of the spectrum at $\nu = 10^{14}\text{Hz}$, but we are in the Wein part of the spectrum at $\nu = 10^{15}\text{Hz}$.

Thus, letting $x = \frac{(\nu F_\nu)_{14}}{(\nu F_\nu)_{15}}$,

$$x = \frac{\nu_{14}\left(2h\nu_{14}^3/c^2\right)\left[e^{h\nu_{14}/kT} - 1\right]^{-1}}{\nu_{15}\left(2h\nu_{15}^3/c^2\right)e^{-h\nu_{15}/kT}} \tag{51}$$

$$= \left(\frac{\nu_{14}}{\nu_{15}}\right)^4 \frac{e^{h\nu_{15}/kT}}{e^{h\nu_{14}/kT} - 1} \tag{52}$$

$$\frac{dx}{dT} \approx x\left(h\nu_{15}/kT^2 + \cdots (\text{ higher order terms})\right) \quad (53)$$

Therefore, $\frac{\Delta x}{x} \approx \frac{h\nu_{15}}{kT}\frac{\Delta T}{T}$. So, $\boxed{\frac{\Delta T}{T} \approx 1\%}$.

7c.

From Kepler's Law:

$$GM_{tot}\left(\frac{P}{2\pi}\right)^2 = a^3 \quad (54)$$

$$\Delta M = \frac{4\pi^2}{GP^2}\left(3a^2\Delta a\right) \quad (55)$$

$$\frac{\Delta M}{M} = 3\frac{\Delta a}{a} = \boxed{33\%} \quad (56)$$

7d.

Finding the Luminosity is straightforward:

$$L = 4\pi R^2 \sigma T_{eff}^4 = 3.9 \times 10^{33}\text{erg/s} = L_\odot \text{ as assumed} \quad (57)$$

Now we calculate the uncertainty:

$$\frac{\Delta L}{L} = \left[\left(\frac{2\Delta R}{R}\right)^2 + \left(\frac{4\Delta T}{T}\right)^2\right]^{1/2} \quad (58)$$

$$\boxed{\frac{\Delta L}{L} \approx 4300\%} \quad (59)$$

7e.

The apparent flux at earth is $F = L/4\pi d^2$. Now, converting to the observed variable Π, we find

$$\frac{\Delta L}{L} = \left[\left(\frac{2\Delta\Pi}{\Pi}\right)^2 + \left(\frac{\Delta F}{F}\right)^2\right]^{1/2} \approx \boxed{21\%} \quad (60)$$

This is a great improvement. Now we do the same, but for $F = R^2\sigma T_{eff}^4/d^2$ to find

$$\frac{\Delta R}{R} = \left[\left(\frac{2\Delta T}{T}\right)^2 + \left(\frac{\Delta\Pi}{\Pi}\right)^2 + \left(\frac{0.5\Delta F}{F}\right)^2\right]^{1/2} \approx \boxed{10.3\%} \quad (61)$$