Solution: HW 2
AY 123, Fall 2007

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Problem 1:

We consider a beam of particles with distribution function $n(\mathbf{p})$ passing through an area element $d\mathbf{A}$. The beam travels at an angle θ to the area element. The number of particles passing through the area per unit time is $(v\cos\theta dA)n(\mathbf{p})\mathbf{dp}$. The momentum delivered per particle is $p\cos\theta$. The total momentum delivered by the beam per second per area is the pressure:

$$dP = pv\cos^2\theta n(\mathbf{p})\mathbf{dp} \tag{1}$$

which must be integrated over all momenta. Taking an isotropic distribution made up of such beams, we get

$$P = 2\pi\int_0^{\pi}\int_0^{\infty} n(p)pv\cos^2\theta\sin\theta p^2 dpd\theta \tag{2}$$

$$\boxed{P = \tfrac{4\pi}{3}\textstyle\int_0^{\infty} n(p)pvp^2dp} \tag{3}$$

Next, for photons use $v = c$ and $n(p) = 2/h^3\left(\exp(pc/kT) - 1\right)$ to find

$$P = \frac{8\pi c}{3h^3}\int_o^{\infty}\frac{p^3}{e^{\frac{pc}{kT}} - 1}dp \tag{4}$$

$$P = \frac{8\pi}{3}\frac{k^4T^4}{h^3c^3}\int_o^{\infty}\frac{x^3}{e^x - 1}dx \tag{5}$$

$$P = \frac{8\pi}{3}\frac{k^4T^4}{h^3c^3}\frac{\pi^4}{15} \tag{6}$$

where $x = pc/kT$ and the definite integral was taken from a table of integrals. This is cleaned up to find

$$\boxed{P = \tfrac{1}{3} a T^4} \tag{7}$$

$$a = \frac{8\pi^5}{15} \frac{k^4}{h^3 c^3} \tag{8}$$

Problem 2:

2a.

We can calculate this as follows. The average number of particles per nucleon is $n = 0.75 * 2 + 0.25 * 3/4 = 1.6875$ and the average mass per nucleon is $m = 0.75 * 1 + 0.25 = 1$. The mean molecular weight is then

$$\boxed{\mu = \tfrac{m}{n} = 0.5926} \tag{9}$$

2b.

Equations (7.28), (7.37), and (7.40) from HKT are combined to find equation (7.41) from HKT:

$$T_c = \frac{1}{(n+1)(-\xi\theta_n')_{\xi_1}} \frac{G\mu}{N_A k} \frac{M}{R} \tag{10}$$

For $\gamma = 5/3$, we have $n = 1.5$. Taking the value for $(-\xi\theta_n')_{\xi_1}$ from table 7.1 in HKT, this gives

$$\boxed{T_c = 1.24 \times 10^7 \mu \tfrac{M}{M_\odot} \tfrac{R_\odot}{R}\ \mathrm{K}} \tag{11}$$

2c.

We assume $R \propto M^{0.08}$. If we assume that this relation holds through a solar mass, then we can replace the proportionality with equality:

$$\frac{R}{R_\odot} = \left(\frac{M}{M_\odot}\right)^{0.08} \tag{12}$$

Inserting this into the expression from **2b** gives

$$\boxed{T_c = 1.24 \times 10^7 \mu \left(\tfrac{M}{M_\odot}\right)^{0.92}\ \mathrm{K}} \tag{13}$$

Similarly, from equation (7.31) in HKT, the central pressure is given by

$$P_c = \frac{1}{4\pi(n+1)(\theta_n')_{\xi_1}}{}^2 \frac{GM^2}{R^4} \tag{14}$$

which with the given mass-radius relation becomes

$$\boxed{P_c = 8.66 \times 10^{15} \left(\tfrac{M}{M_\odot}\right)^{1.68}\ \mathrm{dyne\,cm^{-2}}} \tag{15}$$

2d.

Inserting the pressure and temperature expressions into the result of question 1 gives, with $\mu = 0.6$,

$$\boxed{M = 19 M_\odot} \tag{16}$$

Problem 3:

The value of n corresponding to $\gamma = 1$ is

$$\boxed{n = \infty} \tag{17}$$

Take the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \tag{18}$$

and replace the pressure with $P = K\rho$ to find

$$K\frac{d\rho}{dr} = -\frac{GM\rho}{r^2} \tag{19}$$

Divide both sides by ρ/r^2 and use $\rho = \rho_c \mathrm{e}^{-\psi}$ to get

$$r^2\frac{d\psi}{dr} = \frac{GM}{K} \tag{20}$$

Now differentiate with respect to r and use mass conservation to find:

$$\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = \frac{4\pi G\rho}{K}r^2 \tag{21}$$

Replacing ρ once more with its ansatz and defining $\xi = \sqrt{\frac{4\pi G\rho_c}{K}}r$, we get the requested equation:

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = \mathrm{e}^{-\psi}} \tag{22}$$

It is clear from the definition of ψ that unless ψ approaches ∞ for a finite ξ, the density will never reach zero. This can only happen, though, if the slope of ψ approaches ∞, and so on recursively to higher derivatives. But this is clearly not going to happen from the above differential equation. Hence the density is nonzero for all radii and the star has no edge.

Some examples of objects for which isothermal models are useful are: globular clusters, dark matter halos, galaxies in clusters, hot gas in galaxy clusters, and gas cores in star forming regions.

4:

We know that $U = \frac{2}{3}(\gamma - 1)^{-1} K$, where $K = \frac{3}{2}\int_m \frac{P}{\rho} dm$.

For a polytrope $P \propto \rho^\gamma$, $U \propto K \propto \frac{P}{\rho} \propto \rho^{\gamma-1}$. But $\rho \propto R^{-3}$, so $K = aR^{3(1-\gamma)}$.

In addition, $\Omega \propto R^{-1}$, so the total energy of the star is

$$E_{tot} = K + \Omega = aR^{3(1-\gamma)} - bR^{-1} \tag{23}$$

where $a > 0$ and $b > 0$.

Consider $\gamma = 4/3 + z$. Then $E_{tot} = \frac{aR^{-3z} - b}{R}$. If $z > 0$, then we are are limited to $+\infty$ at small R, 0 at large R, and negative somewhere in the middle for stability. If $z < 0$, then we are are limited to $-\infty$ at small R, 0 at large R, and negative somewhere in the middle for stability. Thus, it is evident that $\gamma > 4/3$ is a concave up stable solution and $\gamma < 4/3$ is a concave down unstable solution. Therefore, $\gamma < 4/3$ is an unstable configuration because a small perturbation in R grows.

5:

The borderline of convective stability is at adiabacity. Beginning with the adiabatic assumption,

$$PT^{1/\gamma-1} = \text{constant} \tag{24}$$

Differentiate with respect to pressure and solve to find

$$\frac{dT}{dP} = (1 - 1/\gamma)\frac{T}{P} \tag{25}$$

Now, we are interested in the temperature profile, so, using the chain rule of calculus:

$$\frac{dT}{dr} = \frac{dT}{dP}\frac{dP}{dr} \tag{26}$$

Plugging the previous derivative and the equation of hydrostatic equilibrium into this equation gives

$$\frac{dT}{dr} = -(1 - 1/\gamma)\frac{T}{P}\frac{GM\rho}{r^2} \tag{27}$$

Replace ρ with the ideal gas equation of state to find

$$\frac{dT}{dr} = -(1 - 1/\gamma)\frac{GM\mu m_H}{kr^2} \tag{28}$$

Inserting $\gamma = 4/3$,

$$\frac{dT}{dr} = -\frac{GM\mu m_H}{4kr^2} \tag{29}$$

We can now integrate inwards from the surface to get:

$$\boxed{T - T_s = \tfrac{GM\mu m_H}{4k}\left(\tfrac{1}{r} - \tfrac{1}{r_s}\right)} \tag{30}$$

6:

6a.

The Planck function is

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\mathrm{e}^{\frac{h\nu}{kT}} - 1} \tag{31}$$

which can be differentiated to give

$$\frac{dB_\nu}{dT} = \frac{2h\nu^3}{c^2}\frac{\frac{h\nu}{kT^2}\mathrm{e}^{\frac{h\nu}{kT}}}{\left(\mathrm{e}^{\frac{h\nu}{kT}} - 1\right)^2} \tag{32}$$

or

$$\boxed{\frac{dB_\nu}{dT} = \frac{2h^2\nu^4}{c^2kT^2}\frac{\mathrm{e}^{\frac{h\nu}{kT}}}{\left(\mathrm{e}^{\frac{h\nu}{kT}} - 1\right)^2}} \tag{33}$$

6b.

Eliminating the frequency through the substitution $x = \frac{h\nu}{kT}$,

$$\boxed{\frac{dB_\nu}{dT} = \frac{2k^3T^2}{h^2c^2}\frac{x^4\mathrm{e}^x}{(\mathrm{e}^x - 1)^2}} \tag{34}$$

6c.

Taking

$$\kappa_\nu\rho = A\frac{\rho^2 g_{ff}}{\nu^3T^{1/2}}\left(1 - \mathrm{e}^{-\frac{h\nu}{kT}}\right) \tag{35}$$

$$\kappa_\nu\rho = A\frac{\rho^2h^3g_{ff}}{k^3T^{7/2}}\frac{\mathrm{e}^x - 1}{x^3\mathrm{e}^x} \tag{36}$$

where $A = 1.32 \times 10^{56}$ cgs. Combining with the results from the previous problem, this gives

$$\boxed{\frac{1}{\kappa_\nu\rho}\frac{dB_\nu}{dT} = \frac{2k^6T^{5.5}}{A\rho^2c^2h^5g_{ff}}\frac{x^7\mathrm{e}^{2x}}{(\mathrm{e}^x - 1)^3}} \tag{37}$$

The active term

$$f(x) = \frac{x^7\mathrm{e}^{2x}}{(\mathrm{e}^x - 1)^3} \tag{38}$$

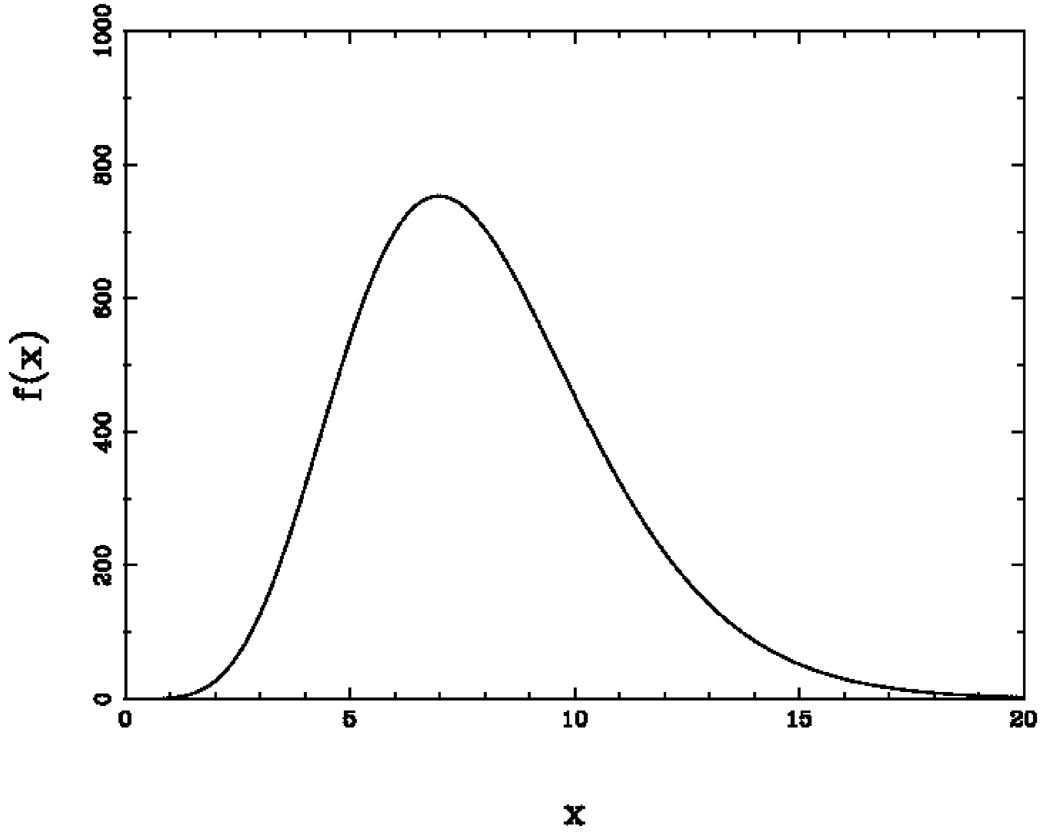

Figure 1: f(x) v. x

is plotted in Figure 1. As is clear, this function is strongest around x of a few, and drops off rapidly elsewhere. Therefore the integral will depend mainly upon its value when x is nearly unity.

6d. The Rosseland mean opacity is calculated from

$$\frac{1}{\kappa} = \frac{\int \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu}{\int \frac{dB_\nu}{dT} d\nu} \tag{39}$$

Inserting the above functions, ignoring constants, and recognizing that the integrals over x are merely constants, we find

$$\frac{1}{\kappa} \propto \frac{T^{5.5}/\rho}{T^2} \tag{40}$$

or

$$\boxed{\kappa \propto \rho T^{-3.5}} \tag{41}$$

7:

7a.

The Saha equation for the single state H atom is:

$$\frac{N(H^-)}{N(H)n_e} = \frac{g(H^-)}{g(H)g_e}\left(\frac{h^2}{2\pi m_e kT}\right)^{3/2} e^{\chi_{H^-}/kT} \tag{42}$$

Fixing the proton's spin direction, the statistical weights will depend upon the electron spin freedom. For a free electron and the neutral hydrogen, there are two electron spin directions. For the H^-, the Pauli exclusion principle locks in the electron spins, so there is only one allowed configuration. Using this along with $P_e = n_e kT$,

$$\boxed{\frac{N(H^-)}{N(H)} = \frac{1}{4}P_e\frac{h^3}{(2\pi m_e)^{3/2}(kT)^{5/2}}e^{\chi_{H^-}/kT}} \tag{43}$$

7b.

The Boltzmann equation is

$$\frac{n_a}{n_b} = \frac{g_a}{g_b}e^{-(E_a-E_b)/kT} \tag{44}$$

For the statistical weights, use $g_a = 2a^2$. For the energies, use $E_a = \chi_H/a^2$. This gives

$$\boxed{\frac{N(H;n{=}2)}{N(H;n{=}1)} = 4e^{-3\chi_H/4kT}} \tag{45}$$

$$\boxed{\frac{N(H;n{=}3)}{N(H;n{=}1)} = 9e^{-8\chi_H/9kT}} \tag{46}$$

7c.

Make the assumption that $N(H) \simeq N(H;n=1)$ so that we can equate the previous two results:

$$N(H^-) = N(H;n=3) \tag{47}$$

$$\frac{1}{4}P_e\frac{h^3}{(2\pi m_e)^{3/2}(kT)^{5/2}}e^{\chi_{H^-}/kT} = 9e^{-8\chi_H/9kT} \tag{48}$$

$$\frac{1}{36}P_e\frac{h^3}{(2\pi m_e)^{3/2}(kT)^{5/2}}e^{(8\chi_H/9+\chi_{H^-})/kT} = 1 \tag{49}$$

The last equation can be solved numerically for $P_e = 10^{1.5}$ dynes to find

$$\boxed{T_{\rm eq} = 7000\,K} \tag{50}$$

Plugging this value back into the Boltzmann equation gives

$$\frac{N(H;n=3)}{N(H;n=1)} = 2.4 \times 10^{-8} \tag{51}$$

justifying our earlier assumption.

7d.

The opacity is given by $\kappa = \sigma n/\rho$ where σ is the cross-section. At low temperatures, the opacity above the Balmer break is due to H^- and below is due to absorbtion by $n = 2$, so:

$$\frac{\kappa(3647^+)}{\kappa(3647^-)} = \frac{\sigma_{H^-}}{\sigma_2}\frac{N(H^-)}{N(H;n=2)} \tag{52}$$

$$\boxed{\frac{\kappa(3647^+)}{\kappa(3647^-)} = \frac{\sigma_{H^-}}{\sigma_2}\frac{1}{16}P_e\frac{h^3}{(2\pi m_e)^{3/2}(kT)^{5/2}}\mathrm{e}^{(3\chi_H/4+\chi_{H^-})/kT}} \tag{53}$$

Whereas at high temperatures, the opacity above the break is due to absorbtion by $n = 3$. Hence

$$\frac{\kappa(3647^+)}{\kappa(3647^-)} = \frac{\sigma_3}{\sigma_2}\frac{N(H;n=3)}{N(H;n=2)} \tag{54}$$

$$\boxed{\frac{\kappa(3647^+)}{\kappa(3647^-)} = \frac{\sigma_3}{\sigma_2}\frac{9}{4}\mathrm{e}^{-5\chi_H/36kT}} \tag{55}$$

Clearly the lower temperature ratio depends on both T and P_e, while the higher temperature ratio depends only on T.