Solution: HW 3
AY 123, Fall 2007

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Problem 1:

Taking after the treatment in HKT, we have energy generation in the form:

$$\epsilon = \epsilon_o \rho^\lambda T^\nu \tag{1}$$

For the CNO cycle, we have $\lambda = 1$ and $\nu = 17$.

The opacity will be of the form:

$$\kappa = \kappa_o \rho^n T^{-s} \tag{2}$$

For Thomson scattering, $n = s = 0$.

The equation of state will have the form

$$P = P_o \rho^{\chi_\rho} T^{\chi_T} \tag{3}$$

We anticipate that from the use of CNO and Thomson that these will be massive stars. We then expect pressure to be due to radiation: $\chi_\rho = 0$ and $\chi_T = 4$. For comparison, we'll also do the exercise for gas pressure: $\chi_\rho = 1$ and $\chi_T = 1$.

HKT (1.70) and (1.73) give

$$R \propto M^{\alpha_R} \tag{4}$$

$$L \propto M^{\alpha_L} \tag{5}$$

with (1.76) and (1.78) giving:

$$\alpha_R = \frac{1}{3}\left[1 - 2(\chi_T + \nu - s - 4)/D_{\mathrm{rad}}\right] \tag{6}$$

$$\alpha_R = 1 + \left[2(\chi_T + \nu - s - 4) - 2\nu(\chi_\rho + \lambda + n)\right]/D_{\mathrm{rad}} \tag{7}$$

where (1.75) has

$$D_{\mathrm{rad}} = (3\chi_\rho - 4)(\nu - s - 4) - \chi_T(3\lambda + 3n + 4) \tag{8}$$

Plugging in the values for radiation pressure gives:

$$\boxed{\alpha_R = 0.475 \text{ and } \alpha_L = 1} \tag{9}$$

whereas gas pressure gives:

$$\boxed{\alpha_R = 0.8 \text{ and } \alpha_L = 3} \tag{10}$$

These stars are marked on an HR diagram in Figure 1

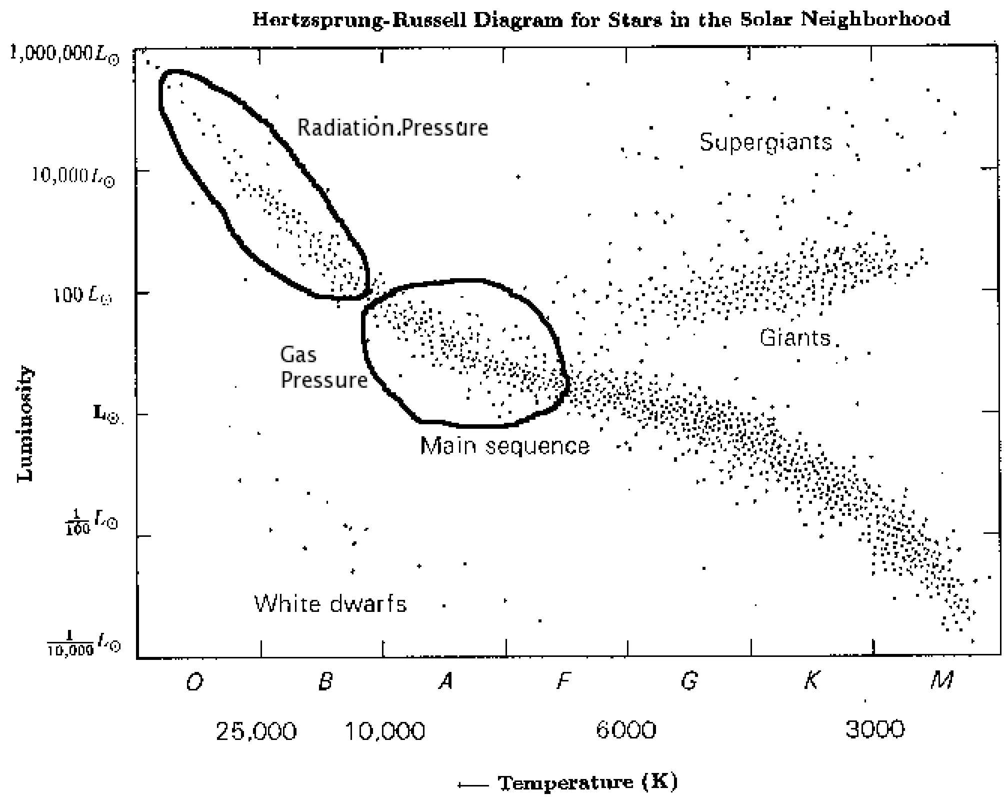


Figure 1: Original taken from ESO.org

Problem 2:

Ionization of hydrogen is derived in HKT in the equations leading up to 3.39. The solution is, for an ionization fraction y,

$$\frac{y^2}{1-y} = \frac{1}{n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-13.6eV/kT} \tag{11}$$

where $n = \rho/m_u$

Now we will try to solve for Γ_1 using the formalism presented in chapter 3.7 of HKT. You are expected to do the derivation of the adiabatic exponents, but it will not be repeated here because it is completed in HKT 3.7.2 with the following results:

$$\chi_\rho = 1 - \mathcal{D}(y) \tag{12}$$

$$\chi_T = 1 + \mathcal{D}(y)\left(\frac{3}{2} + \frac{\chi_H}{kT}\right) \tag{13}$$

$$\Gamma_3 - 1 = \frac{2 + 2\mathcal{D}(y)(3/2 + \chi_H/kT)}{3 + 2\mathcal{D}(y)(3/2 + \chi_H/kT)^2} \tag{14}$$

where

$$\mathcal{D}(y) = \frac{y(1-y)}{(2-y)(1+y)} \tag{15}$$

$$\chi_H = 13.6 \text{ eV} \tag{16}$$

Using 3.98 of HKT, we see that

$$\Gamma_1 = \chi_T(\Gamma_3 - 1) + \chi_\rho \tag{17}$$

$$= \frac{[1 + \mathcal{D}(y)(3/2 + \chi_H kT)]\,[2 + 2\mathcal{D}(y)(3/2 + \chi_H/kT)]}{3 + 2\mathcal{D}(y)(3/2 + \chi_H/kT)^2} + [1 - \mathcal{D}(y)] \tag{18}$$

$$= \frac{\mathcal{D}(y)\,[2(3/2 + \chi_H/kT)^2 + 4(3/2 + \chi_H/kT) - 3] + 5}{2\mathcal{D}(y)(3/2 + \chi_H/kT)^2 + 3} \tag{19}$$

Given ρ, we know y as a function of T, so we know Γ_1 as a funct ion of T, plotted below.

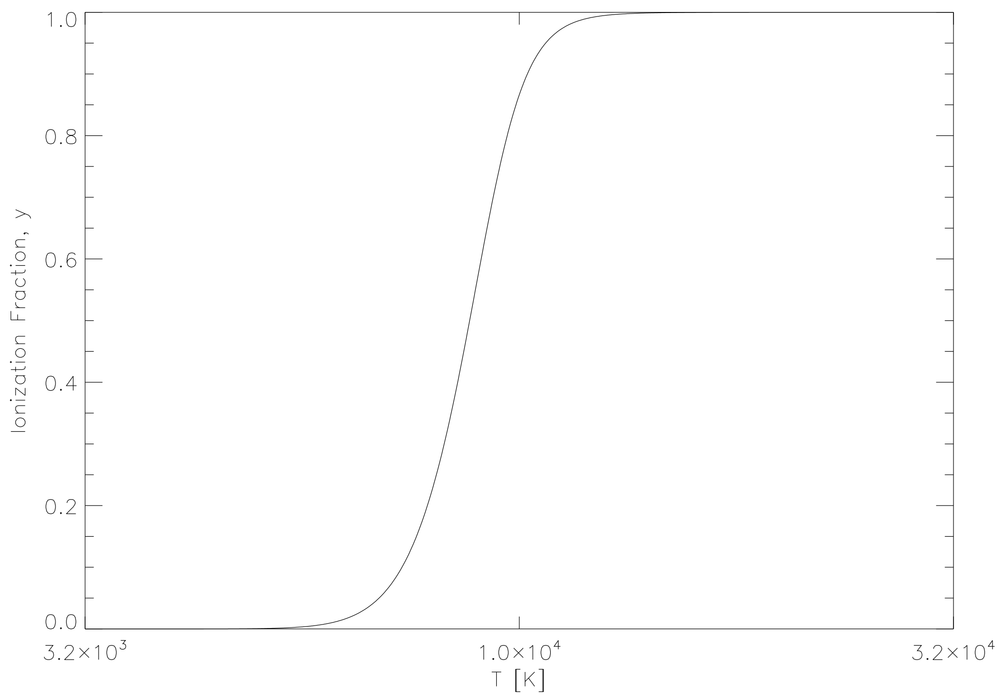


Figure 2: This plot shows the ionization fraction as a function of T.

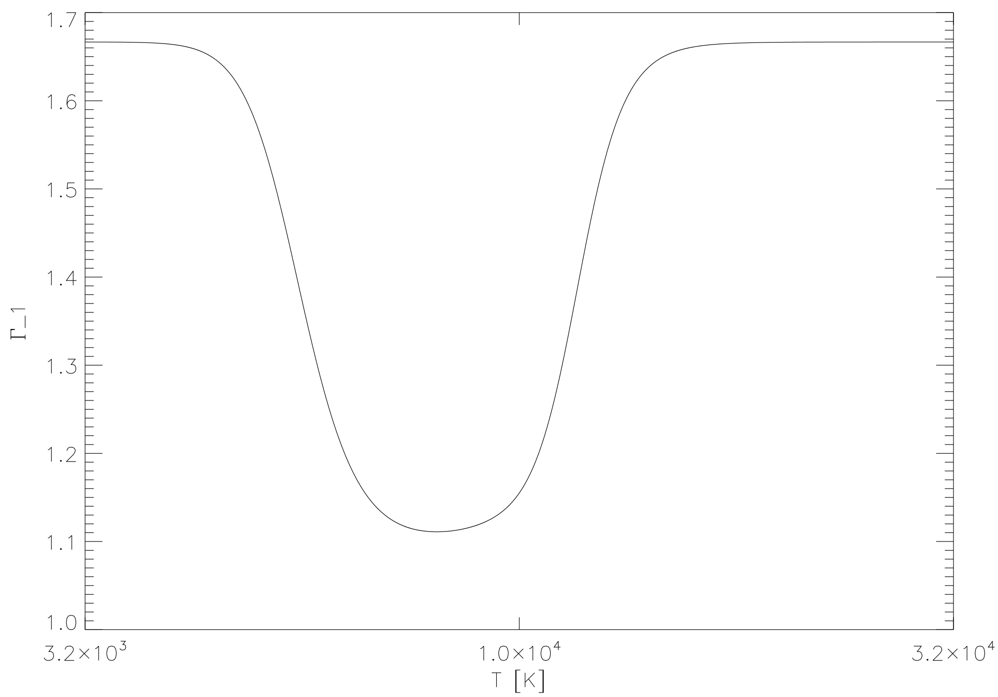


Figure 3: This plot shows Γ_1 as a function of T. Note that Γ_1 goes to 5/3 when there is no ionization or complete ionization, as expected.

Problem 3:

3a.

First we need to find the degeneracy parameters. Since the nuclei are bosons, they can be neglected. We only consider electron spins. For the electron, $g_e = 2$ since the electron can have two spin states relative to some external frame. For neutral helium, the ground state has two electrons in the same s-orbital, so Fermi statistics requires that they have opposite spins; hence the number of allowed electron spin states per atom is $g_0 = 1$. For singly-ionized helium, the electron can assume either spin state (up or down) so that $g_1 = 2$. Finally, for doubly-ionized helium, $g_2 = 1$. The chemical potentials can be determined from the following reaction equations:

$$\mathrm{He}^+ + e^- \leftrightarrow \mathrm{He} + \chi_{\mathrm{He}} \Rightarrow \mu_{\mathrm{He}^+} + \mu_e = \mu_{\mathrm{He}} \tag{20}$$

$$\mathrm{He}^{++} + e^- \leftrightarrow \mathrm{He}^+ + \chi_{\mathrm{He}^+} \Rightarrow \mu_{\mathrm{He}^{++}} + \mu_e = \mu_{\mathrm{He}^+} \tag{21}$$

Therefore, we get

$$n_e = 2\frac{(2\pi m_e kT)}{h^3}\exp(\mu_e/kT)\exp(-m_e c^2/kT) \tag{22}$$

$$n_e = \frac{(2\pi m_0 kT)}{h^3}\exp(\mu_0/kT)\exp(-m_0 c^2/kT) \tag{23}$$

$$n_e = 2\frac{(2\pi m_1 kT)}{h^3}\exp(\mu_1/kT)\exp(-m_1 c^2/kT) \tag{24}$$

$$n_e = \frac{(2\pi m_2 kT)}{h^3}\exp(\mu_2/kT)\exp(-m_2 c^2/kT) \tag{25}$$

Using the μ relations above and $\chi_1 = m_e c^2 + m_1 c^2 - m_0 c^2$, we get

$$\boxed{\frac{n_e n_1}{n_0} = 4\frac{(2\pi m_e kT)}{h^3}\exp(-\chi_1/kT)} \tag{26}$$

and similarly:

$$\boxed{\frac{n_e n_2}{n_1} = \frac{(2\pi m_e kT)}{h^3}\exp(-\chi_2/kT)} \tag{27}$$

3b. Given that $n = n_0 + n_1 + n_2$ and invoking conservation of charge, we get that $z_e n = n_e = n_1 + 2n_2$ or $z_e = z_1 + 2z_2$. We also know that $z_0 + z_1 + z_2 = 1$.

Substituting into the Saha equations:

$$\boxed{\frac{(z_1+2z_2)z_1}{1-z_1-z_2} = 4\frac{(2\pi m_e kT)}{h^3}\exp(-\chi_1/kT)} \tag{28}$$

$$\boxed{\frac{(z_1+2z_2)z_1}{z_1} = \frac{(2\pi m_e kT)}{h^3}\exp(-\chi_2/kT)} \tag{29}$$

3c.

This part just involves plugging in the appropriate T into the above equations and solving for the required values. For n, use $n = \rho/4m_u$ and use the values of ρ given in the problem.

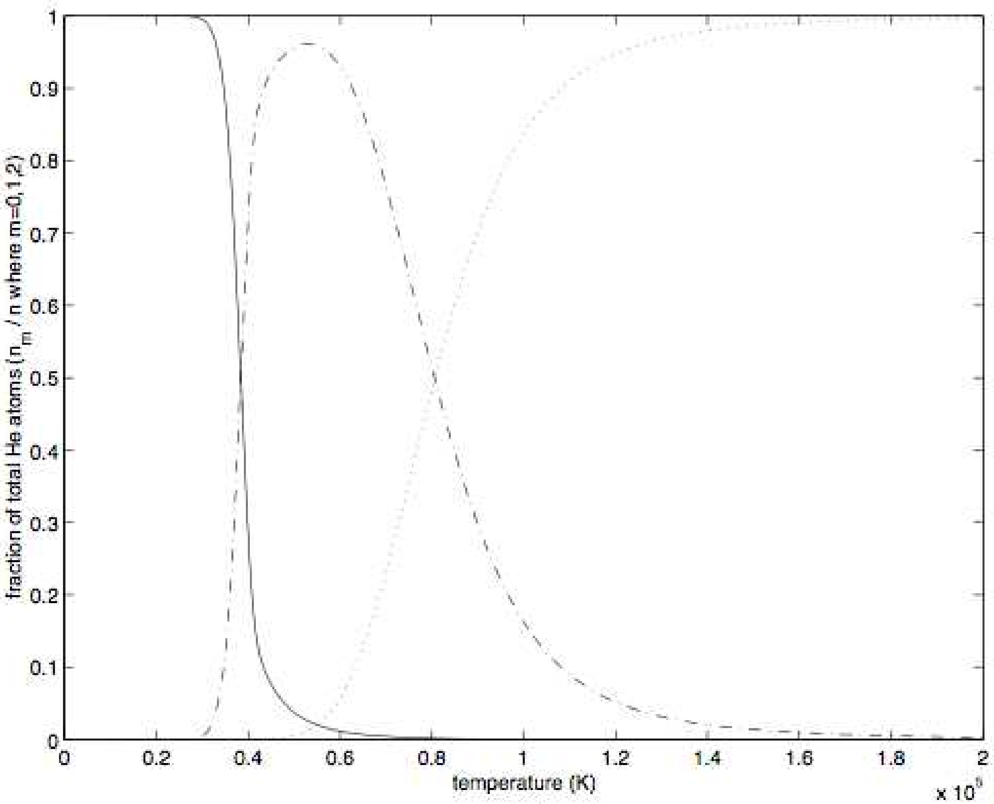
3d.

See the attached figures.

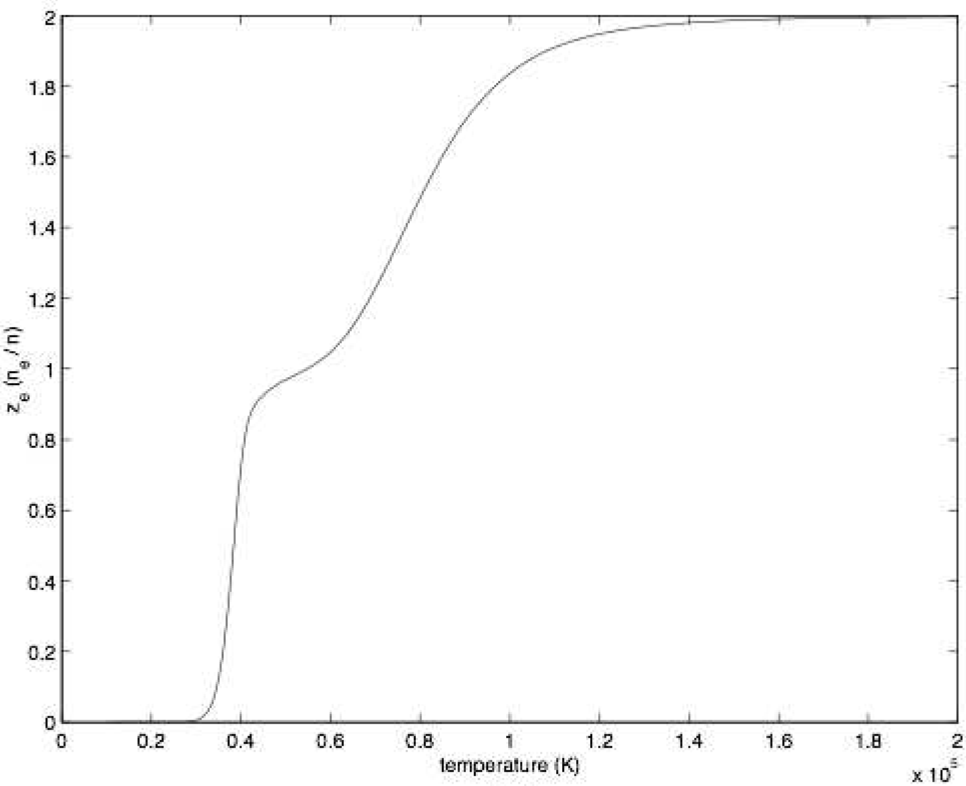
3e.

The temperatures you obtain at the half-ionization points will depend on your assumed density. For $\rho = 10^{-6}\,\mathrm{g\,cm^{-3}}$, the half-ionization points ($z_i = 1/2$) will occur at $T \simeq 35000\,\mathrm{K}$ and $T \simeq 54000\,\mathrm{K}$.

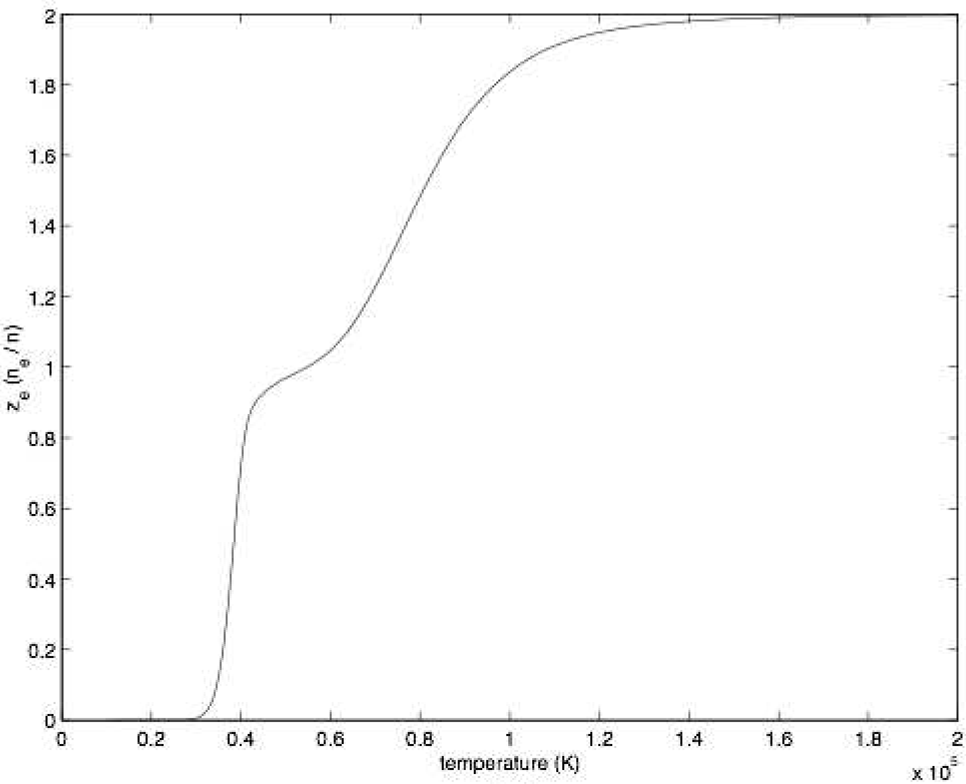
Similarly, for $\rho = 10^{-4}\,\mathrm{g\,cm^{-3}}$, you get the half-ionization points at $T \simeq 38000\,\mathrm{K}$ and $T \simeq 80600\,\mathrm{K}$; for $\rho = 10^{-8}\,\mathrm{g\,cm^{-3}}$, you get the half-ionization points at $T \simeq 25000\,\mathrm{K}$ and $T \simeq 39600\,\mathrm{K}$


fraction of total He atoms (n_m / n where m=0,1,2)
temperature (K)
x 10^5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

$\rho = 10^{-4}$

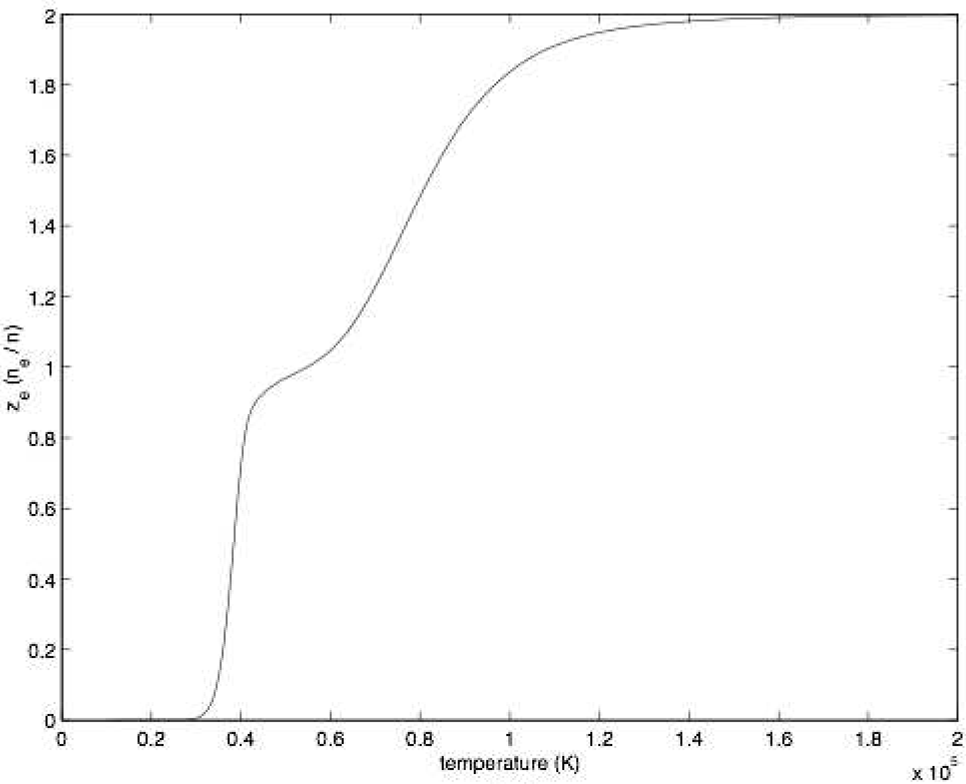


$\rho = 10^{-1}$



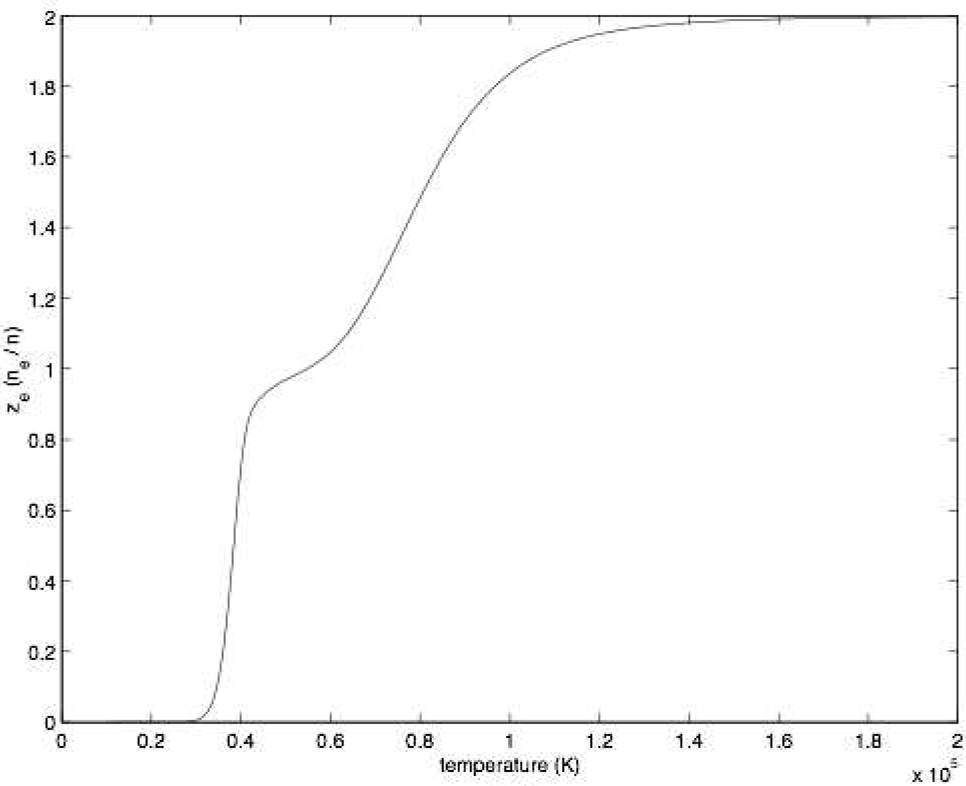
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
temperature (K)
x 10^5
z_e (n_e / n)

$\rho = 10^{-1}$

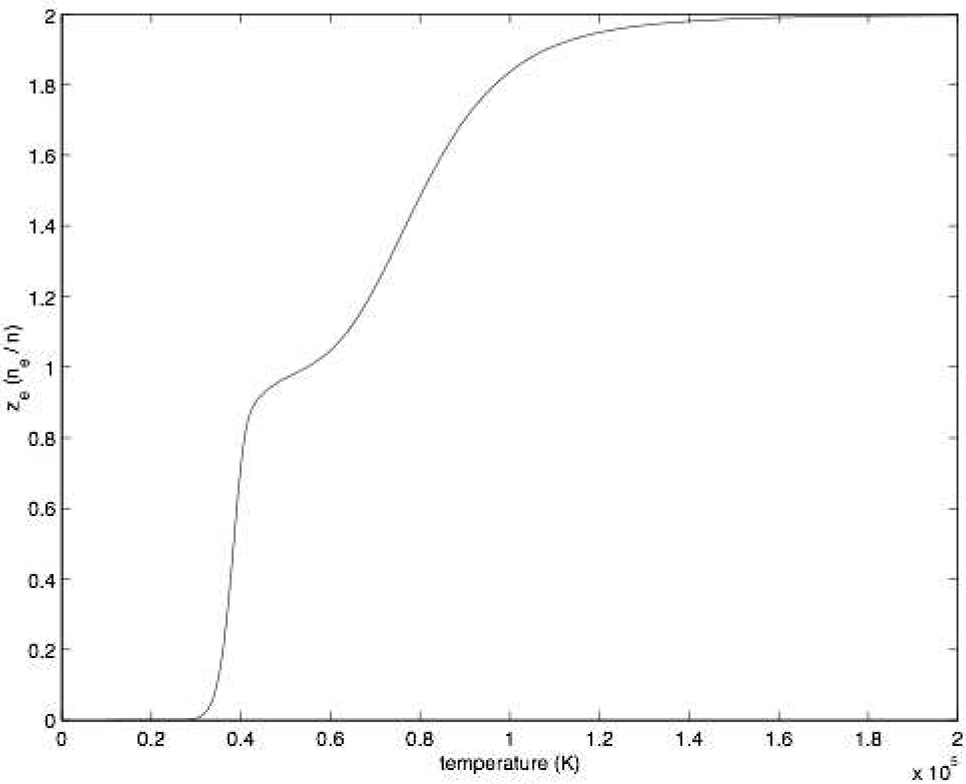


2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
temperature (K)
x 10
z_e (n_e / n)

$\rho = 10^{-1}$



$\rho = 10^{-1}$



2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
z_e (n_e / n)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
temperature (K)
x 10^5

$\rho = 10^{-1}$

Problem 4:

HKT equation (3.66) gives

$$P_e = 1.243 \times 10^{15} \left(\frac{\rho}{\mu_e} \right)^{4/3} \text{ cgs} \tag{30}$$

for a relativistic, degenerate electron gas. This is clearly a polytropic equation of state with $\gamma = 4/3$, or

$$\boxed{n = 3} \tag{31}$$

HKT equation (7.38) gives

$$M = (4\pi)^{-1/2} \left(\frac{n+1}{G} \right)^{3/2} \frac{P_e^{3/2}}{\rho_c^2} \left(-\xi^2 \theta_n' \right)_{\xi_1} \tag{32}$$

From the equation of state,

$$\frac{P_e^{3/2}}{\rho_c^2} = \frac{(1.243 \times 10^{15})^{3/2}}{\mu_e^2} \text{ cgs} \tag{33}$$

Plugging this and the other values into the mass equation with $\mu_e = 2$ for helium,

$$\boxed{M_{\text{chandra}} = 1.4\, M_\odot} \tag{34}$$

Problem 5:

5a.

To recreate the $\rho - T$ phase diagram, lets look at each of the critical lines individually. First, compute the line at which the radiation and (ideal) gas pressure equate.

$$\frac{1}{3}aT^4 = P = \frac{\rho}{\mu m_u}kT \tag{35}$$

$$\rho = \frac{\mu m_u a T^3}{3k} \approx 1.5 \times 10^{-23} T^3 \text{ g/cm}^3 \tag{36}$$

Crystallization lines are determined by the factor Γ_c. This value can be estimated using Lindemann's empirical rule, and is presented in Shapiro and Teukolsky's book Black Holes, White Dwarfs, and Neutron Stars in chapter 4.3. This rough calculation obtains a value of 75, but further experimental work has shown that ion quantum effects play a role and push the value up to ~ 170.

$$\Gamma_c \equiv \frac{(Ze)^2}{r_i kT} = \frac{\text{Coulomb energy}}{\text{Thermal Energy}} \tag{37}$$

where, for ion number density n_i, $\frac{4\pi}{3}r_i^3 \equiv n_i$. Substituting with $n_i = \frac{\rho}{\mu_i m_u}$,

$$\rho = \frac{3\mu_i m_u}{4\pi}\left(\frac{\Gamma_c kT}{(Ze)^2}\right)^3 \tag{38}$$

$$= 8.49 \times 10^{-17} \Gamma_c^3 \mu_i Z^{-6} T^3 \text{ g/cm}^3 \tag{39}$$

Because Γ_c is defined as the ratio of coulomb to thermal energy, we can also use this result with $\Gamma_c > 1$ to find when coulomb forces are important.

Next, consider pressure balance by degeneracy and coulomb effects.

$$P_c = -\frac{3}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{4/3} \tag{40}$$

$$P_{nr} = \frac{3^{2/3}\pi^{4/3}\hbar^2}{5 m_e m_u^{5/3}}\left(\frac{\rho}{\mu_e}\right)^{5/3} \tag{41}$$

Combining the two yields:

$$P_{tot} = P_{nr}\left[1 - \left(\frac{\rho_0}{\rho}\right)^{1/3}\right] \tag{42}$$

with (as will be used in problem 2)

$$\rho_0 = \frac{Z^2 \mu_e m_p}{2\pi^3 a_o^3} \tag{43}$$

$$\approx 0.36 Z^2 \frac{\mu_e}{2} \text{ g/cm}^3 \tag{44}$$

Ionization of H is given by the Saha equation:

$$\frac{y^2}{1-y} = \frac{1}{n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} {-\chi_{\rm H}/\rm kT} \tag{45}$$

and setting y=1/2 to get:

$$\rho = 2m_H \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} {-\chi_{\rm H}/\rm kT} \tag{46}$$

Finally, we will look at degeneracy conditions. Degeneracy is lifted when the Fermi energy of the electrons aquire enough thermal energy to produce a large maxwellian tail. The rough condition for this to occur is $\mathcal{E}_F \approx kT$. So, for non relativistic electrons,

$$kT \approx \mathcal{E}_F = mp_F^2/2 = 4.16 \times 10^{-11} \left(\frac{\rho}{\mu_e}\right)^{2/3} \tag{47}$$

$$\frac{\rho}{\mu_e} \approx 6.0 \times 10^{-9} T^{3/2} \text{ g/cm}^3 \tag{48}$$

However, we need to know when the electrons become relativistic to know where our non-relativistic approximation breaks down. This occurs when $p_F \approx m_e c$, so

$$\frac{\rho}{\mu_e} \approx 9.8 \times 10^5 \text{ g/cm}^3 \tag{49}$$

See the attached figure to see the equations in action.

5b.

In part a, we had

$$\rho_0 = \frac{Z^2 \mu_e m_p}{2\pi^3 a_o^3} \tag{50}$$

$$\approx 0.18 Z^2 \mu_e \tag{51}$$

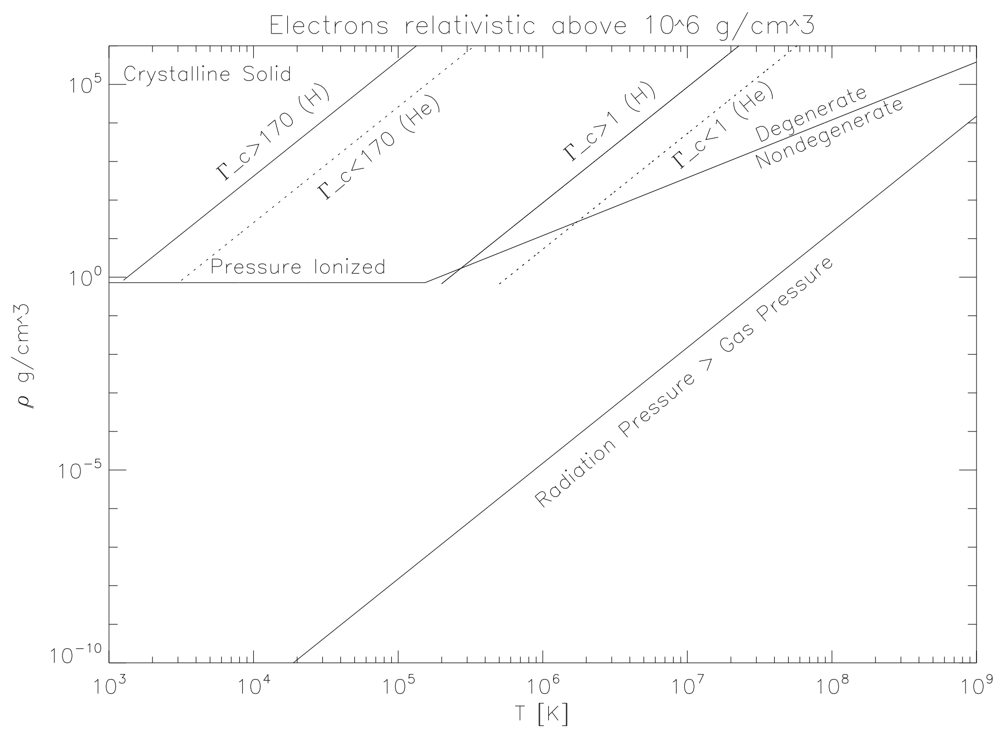


Figure 4: This plot shows all the curves we discussed in the problem.

as the demarcation between coulomb and degeneracy pressures. This is the same as the statement of 1 proton per bohr radius cubed when neglecting constants,

$$\rho \sim m_p/a_0^3 \tag{52}$$

With $\mu_e \sim 2$, we only need Z^2 to be within an order of magnitude of $\pi^3 \approx 31$, which is most of the lighter elements.

5c.

For this problem, we need to look up the densities of liquid hydrogen, liquid helium, water, and iron. These can be found on a number of websites and are presented in the table below. We can also get the volume per atom by $\frac{Am_p}{\rho N_{atoms}}$.

Item	Measured Density[g/cc]	Mass[amu]	Volume per Atom[a_0^3]	separation[a_0]
Liquid Hydrogen	0.0708	1	158	5.4
Liquid Helium	0.125	4	359	7.1
Water	1.0	18	67.2	4.1
Iron	7.86	56	79.8	4.3

Table 1: All values are in g/cm^3

As you can see, the volumes per atom all agree within a factor of a few. The typical atomic separation is ~ 5.2.