Solution: HW 4
AY 123, Fall 2007

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Problem 1:

1a:

To approximate the Gamow peak by a gaussian, first rewrite the $< \sigma v >$ equation as a function of energy. To do so, simply substitute $v = \sqrt{\frac{2E}{m}}$ and $dv = \frac{dE}{\sqrt{2mE}}$. Then, assuming S is an experimentally derived constant near the Gamow peak,

$$< \sigma v >= \left(\frac{8}{\pi m}\right)^{1/2} (kT)^{-3/2} S_0 \int_0^\infty \mathrm{Exp}\left[-\frac{E}{kT} - \frac{b}{E^{1/2}}\right] dE \tag{1}$$

where b is given in the problem.

We wish to compare the integrand to a gaussian,

$$f_{Gauss}(E) = C \times \mathrm{Exp}\left[\frac{(E - E_0)^2}{(\Delta/2)^2}\right] \tag{2}$$

The peak E_0 is given by minimizing $\frac{E}{kT} + \frac{b}{E^{1/2}}$, so

$$\frac{1}{kT} - \frac{b}{2E_0^{3/2}} = 0 \tag{3}$$

$$E_0 = \left(\frac{kTb}{2}\right)^{3/2} \tag{4}$$

Clearly, for the Gaussian to have the same peak value as the Gamow integrand,

$$C = \left(\frac{8}{\pi m}\right)^{1/2} (kT)^{-3/2} S_0 \times \text{Exp}\left[-\frac{E_0}{kT} - \frac{b}{E_0^{1/2}}\right] \tag{5}$$

Now compute the second derivatives at E_0 of the Gamow integrand and the gaussian to get the width (the first derivative at E_0 is zero because it is at a maximum):

$$\begin{aligned} f''_{Gamow}(E_0) &= \frac{8}{\Delta^2}C & (6) \\ f''_{Gauss}(E_0) &= \frac{4E_0^2 - 4kTE_0^{3/2} - 3k^2T^2bE_0^{1/2} + k^2T^2b^2}{-4k^2T^2E_0^3}C & (7) \end{aligned}$$

Substituting in our value of b and equating yeilds:

$$\frac{8}{\Delta^2} = \frac{3}{2kTE_0} \tag{8}$$

So,

$$\Delta = \frac{4}{\sqrt{3}}\sqrt{kTE_0} \tag{9}$$

1b:

From our calculations in part a,

$$\begin{aligned} <\sigma v> &= C\int_0^\infty \text{Exp}\left[\frac{(E-E_0)^2}{(\Delta/2)^2}\right] dE & (10) \\ &\propto \left(\frac{T^{-3/2}}{m^{1/2}}\right) S_0 \text{Exp}\left[-\frac{E}{kT} - \frac{b}{E^{1/2}}\right] (\Delta) \int_{-\infty}^{\infty} e^{-x^2} dx & (11) \\ &\propto \left(\frac{T^{-1}E_0}{m^{1/2}}\right) S_0 \text{Exp}\left[-\frac{E_0}{kT} - \frac{b}{E_0^{1/2}}\right] & (12) \\ &\propto \left(\frac{T^{-1}E_0^{1/2}}{m^{1/2}}\right) S_0 \text{Exp}\left[-\frac{3E_0}{kT}\right] & (13) \end{aligned}$$

So, let $\tau = \frac{3E_0}{kT}$, then

$$<\sigma v> \propto \left(\frac{T^{-1}E_0^{1/2}}{m^{1/2}}\right) S_0 e^{-\tau} \times \tau^2 \times \left(\frac{kT}{E_0}\right)^2 \tag{14}$$

$$\propto \left(\frac{TE_0^{-3/2}}{m^{1/2}}\right) S_0 e^{-\tau}\tau^2 \tag{15}$$

$$\propto \left(\frac{T}{bTm^{1/2}}\right) S_0 e^{-\tau}\tau^2 \tag{16}$$

$$\propto \frac{1}{Z_A Z_B m} S_0 \tau^2 e^{-\tau} \tag{17}$$

Problem 2:

2a:

By HKT 6.76, the energy generation rate for the $p + p$ reaction is:

$$\epsilon \propto \frac{\rho}{T_9^{-2/3}} e^{-aT_9^{-1/3}} \qquad a = 3.380 \tag{18}$$

So, $n = 1$. To find η,

$$\eta = \left(\frac{\partial \ln r}{\partial \ln T} \right) \tag{19}$$

$$= \frac{a}{3T_9^{1/3}} - \frac{2}{3} \tag{20}$$

$$= \frac{\tau}{3} - \frac{2}{3} \tag{21}$$

For solar temperature, $\eta \approx 3.9$.

2b:

We are looking at the reaction

$$^7Be + p \ \rightarrow \ ^8B + \gamma, \tag{22}$$

which begins the PP-III chain. The reaction rate for this reaction is

$$r_{^7Be\gamma} = \frac{\rho^2}{m_u^2} \frac{X_{^7Be} X_p}{A_{^7Be} A_p} < \sigma v >_{^7Be\gamma} \text{cm}^{-3} \ s^{-1} \tag{23}$$

The temperature dependence comes in from the $< \sigma v >$:

$$r \ \propto \ < \sigma v >_{^7Be\gamma} \ \propto \ \frac{e^{-aT_6^{-1/3}}}{T_6^{2/3}}, \tag{24}$$

with

$$a = 42.49(Z_{^7Be}^2 Z_p^2 \mu)^{1/3} \ \ and \ \ \mu = \frac{A_{^7Be} A_p}{A_{^7Be} + A_p}. \tag{25}$$

We write the rate as

$$r \propto T^\nu \tag{26}$$

and thus we can solve for ν:

$$\nu = \left(\frac{\partial \ln r}{\partial \ln T}\right) = \frac{a}{3T_6^{1/3}} - \frac{2}{3} \tag{27}$$

We have the following values: $A_{^7Be} = 7$, $A_p = 1$, $Z_{^7Be} = 4$ and $Z_p = 1$. We estimate μ as $\mu = 7/8 \approx 1$. Thus at a temperature of $T = 1.5 \times 10^7 = 15 \times 10^6$K, we have:

$$\nu \approx \frac{42.49(16)^{1/3}}{3(15)^{1/3}} - \frac{2}{3} = 13.8 \approx 14 \tag{28}$$

So we indeed find that the reaction rate is approximately proportional to T^{14}.

How much would the centeral temperature of the sun need to change to solve the solar nuetrino problem? The PP-III chain continues as follows:

$$^8B \;\rightarrow\; ^8Be + e^+ + \nu_e \tag{29}$$

creating an electron nuetrino. Assuming the rate of the chain is dominated by the first reaction (^8B has a half life of 0.8 s), the rate of production of nuetrinos from this chain is $\propto T^{14}$. Now, Davis' experiment can only detect nuetrinos from the PP-II and PP-III chains, and the majority are expected to come from this B inverse beta decay in PP-III. So let's assume that all the neutrinos detected are from the above reaction. The observed nuetrino flux (from HK Section 8.3) is 2.07 ± 0.3 SNU, while the predicted flux is 7.9 ± 2.4 SNU. So to explain the discrepency we will need to lower the predicted number of nuetrinos produced by approximately 4:

$$r_2 \approx \frac{1}{4} r_1 \quad \Rightarrow \quad T_2^{14} \approx \frac{1}{4} T_1^{14} \tag{30}$$

$$\Rightarrow T_2 \approx 0.9 T_1 \tag{31}$$

So we would have to lower the central temperature of the sun by $\sim 10\%$, or $\sim 1.5 \times 10^6$K. Such a change would require a drastic reworking of stellar models. Instead, it is explained through neutrino oscillations, in which part of the electron neutrinos produced change into μ or τ neutrinos on the way to earth. These are not detected by the Ray Davis experiment.

Problem 3:

We are concerned with the PP-I chain, given as follows:

$$p + p \rightarrow d + e^+ + \nu_e \quad (32)$$
$$d + p \rightarrow {}^3He + \gamma \quad (33)$$
$${}^3He + {}^3He \rightarrow {}^4He + p + p \quad (34)$$

The reaction rate for the reaction $\alpha + X \rightarrow Y + \beta$ is given by (e.g. HKT)

$$R_{\alpha\beta} = \frac{1}{(1 + \delta_{\alpha X})} n_\alpha n_X < \sigma v >_{\alpha\beta}, \quad (35)$$

the units of this rate being cm^{-3} s^{-1} (Note that there is a degeneracy factor of 2 if α and X are the same element. $< \sigma v >$ is given by $< \alpha\beta > \equiv < \sigma v >_{\alpha\beta}$, which we obtain from (CF1988) at $T_{c,\odot} = 1.5 \times 10^7$. We let:

$$C_1 = \quad < p, e^+ >= \quad 1.36 \times 10^{-43} \quad (36)$$
$$C_2 = \quad < p, \gamma >= \quad 2.19 \times 10^{-26} \quad (37)$$
$$C_3 = \quad <^3 He, 2p >= \quad 3.89 \times 10^{-34} \quad (38)$$

The rate of change of any given species in the PP chain is given by the rate at which it is created minus the rate at which it is destroyed ($R_{created} - R_{\mathrm{destroyed}}$). The reaction rates for the three steps in the PP chain are given by Eq. 35:

$$R_{pe^+} = \frac{n_p^2}{2} < p, e^+ > \quad (39)$$
$$R_{p\gamma} = n_p n_d < p, \gamma > \quad (40)$$
$$R_{^3He,2p} = \frac{n_{^3He}^2}{2} <^3 He, 2p > \quad (41)$$
$$(42)$$

The above PP reactions give four equations for the abundances of H, D, ^{3}He and ^{4}He:

$$\frac{dn_p}{dt} = -2\left(\frac{n_p^2}{2} < p, e^+ >\right) - n_p n_d < p, \gamma > +2\left(\frac{n_{^3He}^2}{2} <^3 He, 2p >\right) \quad (43)$$

$$\frac{dn_d}{dt} = \frac{n_p^2}{2} < p, e^+ > -n_p n_d < p, \gamma > \tag{44}$$

$$\frac{dn_{^3He}}{dt} = n_p n_d < p, \gamma > -2\left(\frac{n_{^3He}^2}{2} <^3 He, 2p >\right) \tag{45}$$

$$\frac{dn_{^4He}}{dt} = \frac{n_{^3He}^2}{2} <^3 He, 2p > \tag{46}$$

The $< \sigma v >$'s are constant in time, dependent only on temperature and the S_0's. In order to find the time when the first reaction rate Eq. 39 becomes longer than the other two, we need the abundances of of H, D, and ^{3}He as a function of time (and while we're at it we'll calculate the abundance of ^{4}He too as we need to plot it later). To find these abundances we will have to solve the system of differential equations Eqs. 43-46.

If we replace all the n's with $n_i = \rho X_i / A_i m_u$ so that things can be written in terms of ratios X_n, the equations we need to simultaneously solve are:

$$\frac{dX_p}{dt} = \left(\frac{\rho}{m_u}\right)\left[-C_1 X_p^2 - \frac{C_2}{2} X_p X_d + \frac{C_3}{9} X_{^3He}^2\right] \tag{47}$$

$$\frac{dX_d}{dt} = \left(\frac{\rho}{m_u}\right)\left[C_1 X_p^2 - C_2 X_p X_d\right] \tag{48}$$

$$\frac{dX_{^3He}}{dt} = \left(\frac{\rho}{m_u}\right)\left[\frac{3C_2}{2} X_p X_d - \frac{C_3}{3} X_{^3He}^2\right] \tag{49}$$

$$\frac{dX_{^4He}}{dt} = \left(\frac{\rho}{m_u}\right)\left[\frac{2C_3}{9} X_{^3He}^2\right] \tag{50}$$

$$\tag{51}$$

subject to the constraint that $X_p + X_d + X_{^3He} + X_{^4He} = 1$ at all times, and the boundary conditions $X_p(0) = 1$ and $X_d(0) = X_{^3He}(0) = X_{^4He}(0) = 0$, which apply to a star initially composed of pure hydrogen. Our constraint is satisfied by the initial conditions because $\frac{d}{dt}(X_p + X_d + X_{^3He} + X_{^4He}) = 0$. Simplifying:

$$\frac{dX_p}{dt} = -\alpha X_p^2 - \beta X_p X_d + \gamma X_{^3He}^2 \tag{52}$$

$$\frac{dX_d}{dt} = \alpha X_p^2 - 2\beta X_p X_d \tag{53}$$

$$\frac{dX_{^3He}}{dt} = 3\beta X_p X_d - 3\gamma X_{^3He}^2 \tag{54}$$

$$\frac{dX_{^4He}}{dt} = 2\gamma X_{^3He}^2 \tag{55}$$

$$\tag{56}$$

where $\alpha = 1.19 \times 10^{-17}$, $\beta = 0.964$, $\gamma = 3.80 \times 10^{-9}$.

We now want to find the equilibrium timescale. This occurs when deuterium production (the fastest) saturates. Also, during these early times, X_p remains constant at about 1. So, we have,

$$\frac{dX_d}{dt} = \alpha - 2\beta X_d \tag{57}$$

which has the solution

$$X_d(t) = \frac{\alpha}{2\beta}\left(1 - e^{-2\beta t}\right) \tag{58}$$

so it has a timescale $(2\beta)^{-1} \approx 0.5$ seconds and saturates at a value of $\frac{\alpha}{2\beta} \approx 6.17 \times 10^{-18}$. Now we use this saturation value in the third equation to see when H and D come into equilibrium with ^{3}He.

$$\frac{dX_{^3He}}{dt} = \frac{3\alpha}{2} - 3\gamma X_{^3He}^2 \tag{59}$$

which has the solution

$$X_{^3He}(t) = \sqrt{\frac{\alpha}{2\gamma}} tanh\left[\frac{3\sqrt{\alpha\gamma}}{\sqrt{2}}t\right] \tag{60}$$

so it has a timescale $\left(\frac{3}{\sqrt{2}}\sqrt{\alpha\gamma}\right)^{-1} \approx 70249$ years and saturates at a value of $\frac{\alpha}{2\gamma} \approx 1.57 \times 10^{-9}$. Below we plot the X_p and $X_{^3He}$ fractions for the first million years, and the all the abundances over 10^{10} years.

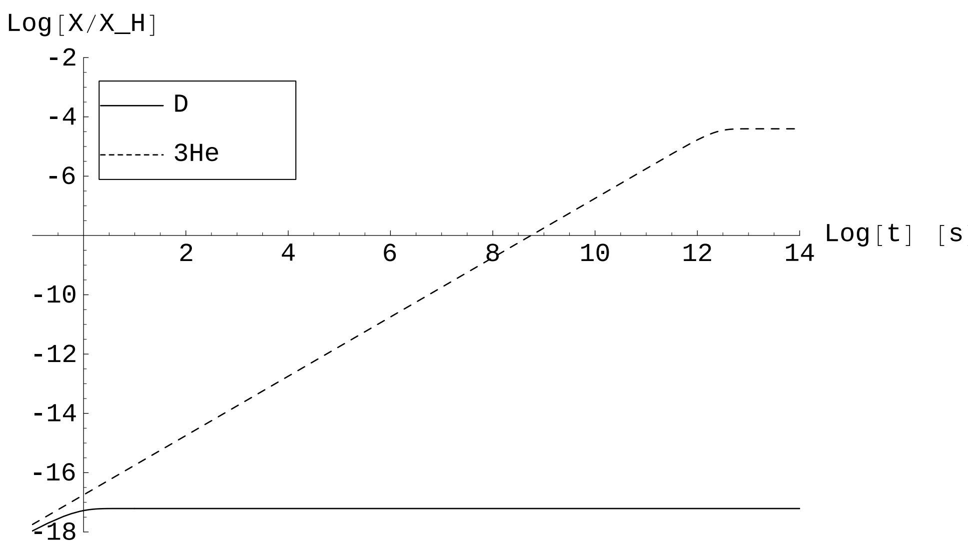

Figure 1: This plot shows the X_p and $X_{^3He}$ fractions for the first 3 years. The solid line, deuterium, comes into equilibrium at abou 0.5 seconds and saturates at a value of 6.17×10^{-18}. The dashed line, helium 3, takes closer to 70 kyr to reach equilibrium and saturates at a value of 3.96×10^{-5}

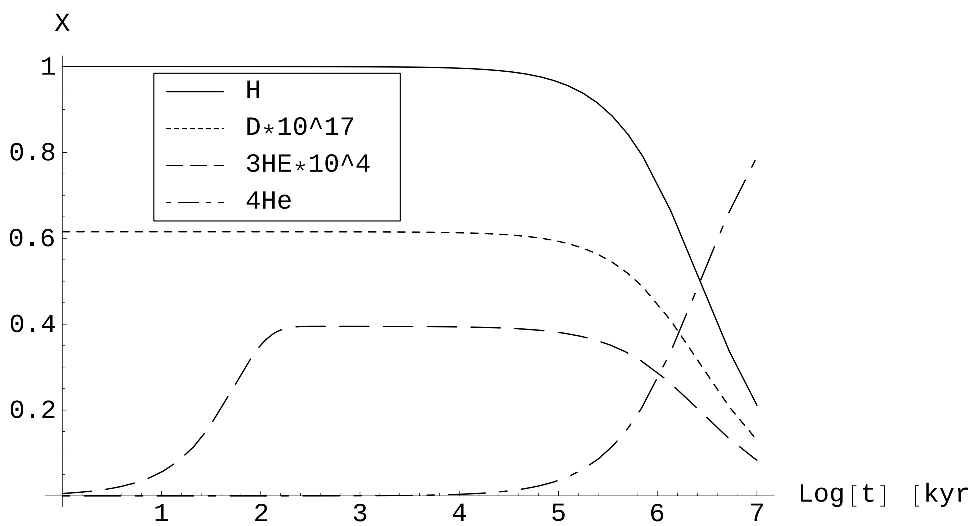

Figure 2: Over the next 10 Gyr, things build up and change, yielding ^{4}He. So, long timescales($\sim$10 Myr) will upset the equilibrium set up on short timescales ($\sim$70 kyr). Note that the D and ^{3}He fractions have been multiplied by a constant factor so they will show up on the graph.

Problem 4:

4a:

In problem set #1, we arrived derived the scaling solutions for ρ_c and T_c. Using solar units for M and R, and the emperical scaling relation $R \propto M^{3/4}$,

$$\begin{aligned} \rho_c &\approx 5.57M^{-5/4} &(61)\\ T_c &\approx 7\times 10^6 M^{1/4} &(62)\end{aligned}$$

Now, using solar values of $X = .7$ and $Z = 0.02$, we find that HKT 6.76-6.77 reduce to

$$\begin{aligned} \epsilon_{pp} &\approx \frac{2.239\times 10^6}{M^{17/12}} e^{-19.766M^{-1/12}} &(63)\\ \epsilon_{CNO} &\approx \frac{1.173\times 10^{26}}{M^{17/12}} e^{-89.405M^{-1/12}} &(64)\end{aligned}$$

4b:

The fracion of energy comming from the pp chain is ϵ_{pp}/ϵ. Similarly for the fraction of energy produced by the CNO cycle. The plots comparing them to the total and each other follow.

4c:

We would like to know when the CNO cycle saturates. This occures when all the protn capture rates are faster than the slowest decay rate in the chain. Looking at the class notes, ^{13}N(e^+,ν_e)^{13}C is the slowest beta decay at 7 minuts. Now we use CF88 to look at what temperatures the other timescales ($(\rho X f_{CF88})^{-1}$) are shorter than this. Therefore, we need to find where $f_{CF88} > 3.4\times 10^{-9}$ for each reaction and take the largest temperature:

^{12}C(p,γ)^{13}N $\Rightarrow$ $T_{crit,9} \sim 0.047$ ^{13}C(p,γ)^{14}N $\Rightarrow$ $T_{crit,9} \sim 0.040$

^{14}N(p,γ)^{15}O $\Rightarrow$ $T_{crit,9} \sim 0.060$ ^{15}N(p,α)^{12}C $\Rightarrow$ $T_{crit,9} \sim 0.031$

So, $T_{sat} = 6\times 10^7$ K.

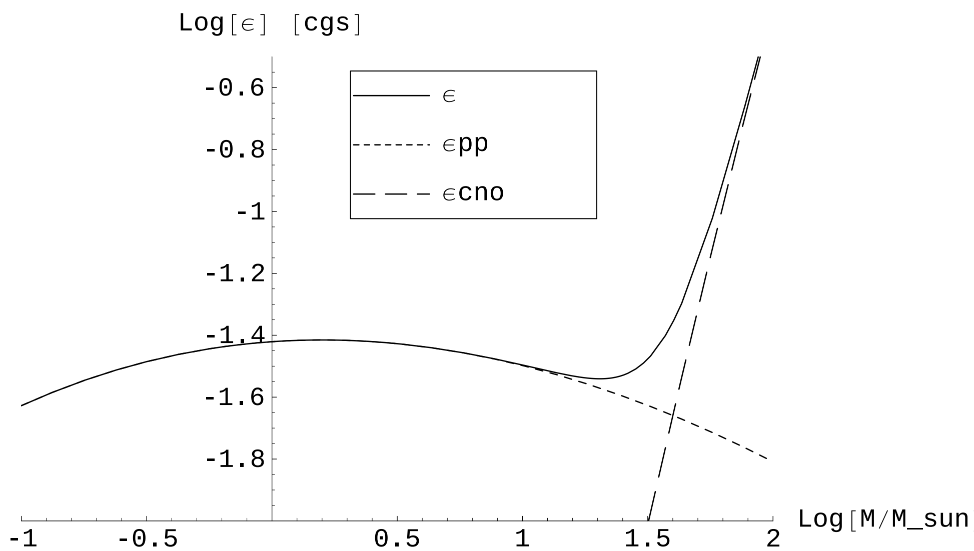

Figure 3: The transition from pp to cno in our model star occurs at a slightly larger mass than in real stars, which are more centrally concentrated than our model.

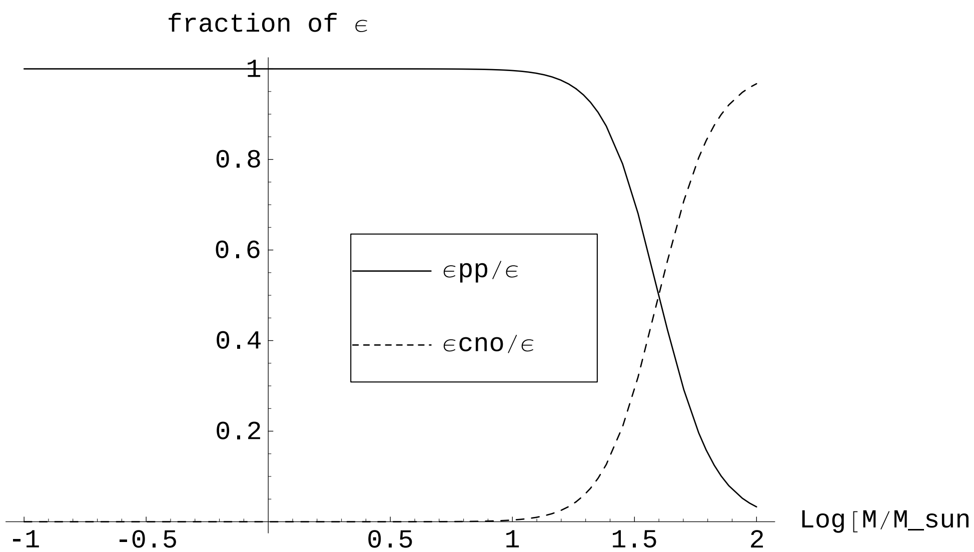

Figure 4: The fraction of energy produced by ϵ_{pp} and ϵ_{CNO}

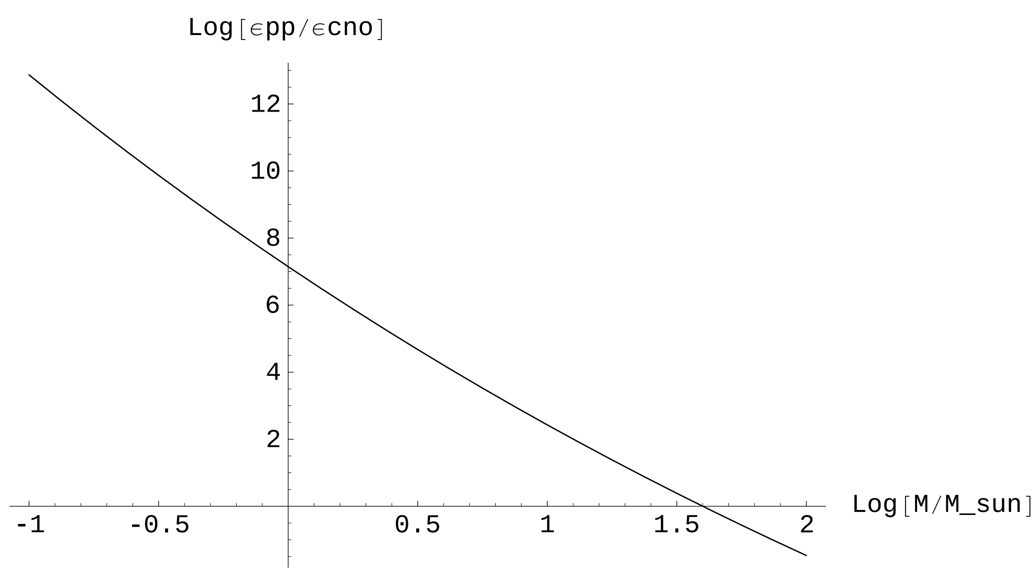

Figure 5: The fraction of energy produced by ϵ_{pp} comparted to ϵ_{CNO}

Problem 5:

The class notes for nuclear reactions give the following equation for the energy generation rate of the triple-α process:

$$\epsilon_{3\alpha} = \frac{5.1 \times 10^8 \rho^2 Y^3}{T_9^3} \mathrm{e}^{-4.40/T_9} \, \mathrm{ergs\, g^{-1}\, s^{-1}} \tag{65}$$

Evaluating this for $\rho = 10^5 \, \mathrm{g\, cm^{-3}}$, $T = 10^8 \, \mathrm{K}$, and $Y = 1$ (pure helium):

$$\boxed{\epsilon_{3\alpha} = 397 \, \mathrm{ergs\, g^{-1}\, s^{-1}}} \tag{66}$$

Now, if the excitation energy of ${}^{12}C^{*}$ is 7.664 MeV instead of 7.654 MeV, then the exponent of $-4.40/T_9$ will change. The 4.40 comes from the mass difference between ${}^{12}C^{*}$ and the three α's. From Fig. 6.11 of HKT, the true $\Delta m = 0.3818 \, \mathrm{MeV}$, which our hypothetical value is $\Delta m = 0.3918 \, \mathrm{MeV}$, an increase of 1.0262 times. The 4.40 then becomes 4.52. We then find:

$$\boxed{\epsilon_{3\alpha} = 119 \, \mathrm{ergs\, g^{-1}\, s^{-1}}} \tag{67}$$

Problem 6:

For this problem, all units are cgs. We are looking at 1 gm of material, so the energy generation rate per unit volume per density, for $V\rho = 1$ gm,

$$\frac{dE}{dt} = \epsilon = 5.1 \times 10^8 \frac{\rho^2 Y^3}{T_9^3} e^{-4.4027/T_9} \tag{68}$$

where

$$dE = c_{V_\rho} dT \tag{69}$$

and the specific heat is the sum of the (degenerate) electron and ideal (ion) specific heats

$$c_{V_\rho} = 1.35 \times 10^5 \frac{T}{\rho} x \left(1 + x^2\right)^{1/2} + \frac{3k}{2\mu_I m_u} \tag{70}$$

and x is the normalized fermi momentum

$$x^3 = \frac{\rho}{B\mu_e} \tag{71}$$

For our ionized helium core,

$$\begin{aligned} \rho &= 2 \times 10^5 && (72) \\ T &= 1.5 \times 10^8 && (73) \\ Y &= 1 && (74) \\ \mu_I &= 4 && (75) \\ \mu_e &= 2 && (76) \\ & && (77) \end{aligned}$$

Putting this all together and converting time into days yields

$$\begin{aligned} \frac{dT_9}{dt_d} &= \frac{\epsilon \times 24 \times 3600}{c_{V_\rho} \times 10^9} && (78) \\ &= \frac{5.05 \times 10^6 e^{-\frac{4.4027}{T_9}}}{T_9^3 \left(T_9 + 0.0893\right)} && (79) \end{aligned}$$

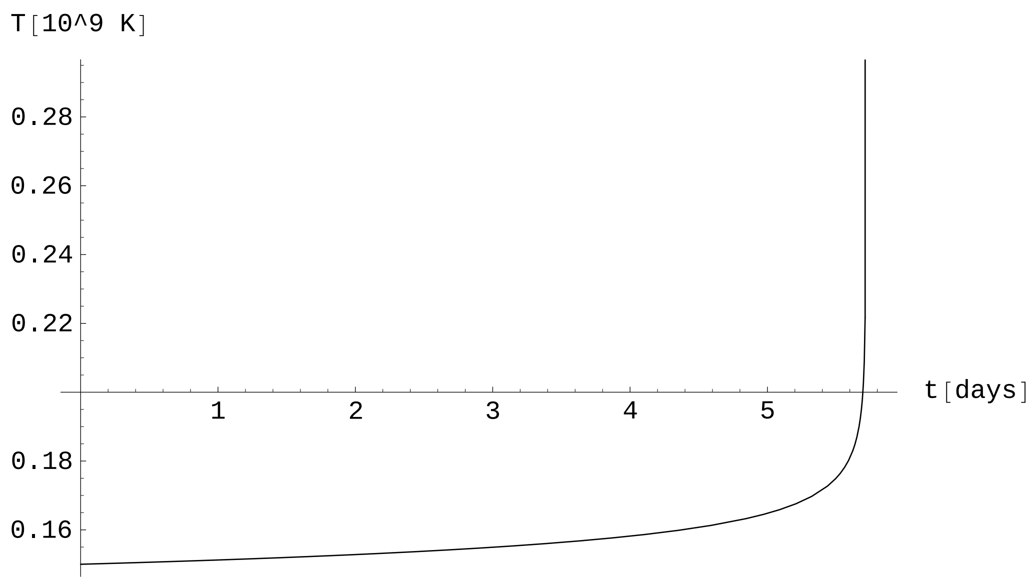

Figure 6: As is seen in the graph, the Helium Flash occurs in about 5 to 6 days. This time, however, is very sensitive to the intial conditions.