Solution: HW 5
AY 123, Fall 2007

by Fabio Altenbach, Melissa Enoch, and Matthew Stevenson

December 2, 2007

Problem 1:

The scale height of an atmosphere is

$$H = \frac{P}{g\rho} \tag{1}$$

If we assume an ideal equation of state:

$$P = \frac{\rho}{\mu} N_a k T \tag{2}$$

then we find

$$H = \frac{N_a k T}{\mu g} \tag{3}$$

The acceleration due to gravity at 500 km above the Sun is:

$$g = \frac{GM_\odot}{R_\odot^2} = 2.73 \times 10^4\,\mathrm{cm\,s^{-2}} \tag{4}$$

This into 3, with $\mu = 1.26$ gives

$$\boxed{H = 100\,\mathrm{km}} \tag{5}$$

We approximate

$$\omega_c \simeq \frac{c_s}{2H} \tag{6}$$

which, with $c_s^2 = \gamma P/\rho$ and the equation for scale height, gives

$$\omega_c = \frac{1}{2}\sqrt{\gamma \frac{g}{H}} \tag{7}$$

Plugging in the previously calculated values, along with $\gamma = 5/3$, gives

$$\boxed{\omega_c = 0.034\,\mathrm{s}^{-1}} \tag{8}$$

The corresponding period is

$$\boxed{T = \tfrac{2\pi}{\omega} = 3\,\mathrm{min}} \tag{9}$$

Problem 2:

Consider a mass element displaced upwards in a star. The element with expand adiabatically; as it does so, its density will change by $\delta\rho$. The surrounding gas, on the other hand, will have a density difference of $\Delta\rho$ from the element's initial position. The force per unit mass on the element will then be buoyancy minus gravity:

$$F = \frac{1}{\rho}\left(g(\rho + \Delta\rho) - g(\rho + \delta\rho)\right) \tag{10}$$

or

$$F = \frac{g}{\rho}\left(\Delta\rho - \delta\rho\right) \tag{11}$$

For small displacements, we can rewrite this:

$$\boxed{F = -N^2\Delta x = \tfrac{g}{\rho}\left(\tfrac{\Delta\rho}{\Delta x} - \tfrac{\delta\rho}{\Delta x}\right)dx} \tag{12}$$

where

$$\boxed{N^2 = -\tfrac{g}{\rho}\left(\tfrac{\Delta\rho}{\Delta x} - \tfrac{\delta\rho}{\Delta x}\right)} \tag{13}$$

is the Brunt-Vaisala frequency.

2a:

If we consider the motion to be adiabatic, then we must conserve $P\rho^{-\gamma}$. This implies that

$$\frac{\delta\rho}{\rho} = \frac{1}{\gamma}\frac{dP}{P} \tag{14}$$

If we assume that the surrounding gas is ideal, then

$$P \propto \rho T \tag{15}$$

so

$$\frac{\Delta\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T} \tag{16}$$

Plugging these into the Brunt-Vaisala equation and letting $\Delta x \to dx$,

$$N^2 = -g\left(\frac{dP}{P} - \frac{dT}{T} - \frac{1}{\gamma}\frac{dP}{P}\right) \tag{17}$$

or

$$\boxed{N^2 = -g\left(\tfrac{\gamma-1}{\gamma}\tfrac{dP}{P} - \tfrac{dT}{T}\right)} \tag{18}$$

2b:

An adiabatic temperature gradient will satisfy

$$PT^{\gamma/(1-\gamma)} = \text{constant} \tag{19}$$

This implies

$$\frac{dT}{T} = \frac{\gamma-1}{\gamma}\frac{dP}{P} \tag{20}$$

A quick comparison with equation 18 shows that an adiabatic temperature gradient implies

$$\boxed{N^2 = 0} \tag{21}$$

This confirms that gravity waves driven by buoyancy oscillations cannot propagate in convective regions of stars.

Problem 3:

The Brunt-Väisälä freqency is given by:

$$N^2 = -\frac{\chi_T}{\chi_\rho}\left(\nabla - \nabla_{ad}\right) g \lambda_p^{-1} \tag{22}$$

A simple planetary atmosphere can be treated as isothermal, so $\nabla = 0$, because there is no temperature gradiant. Also, atmospheric gas is ideal, so $\chi_T = 1$, $\chi_\rho = 1$, and $\nabla_{ad} = \frac{\gamma-1}{\gamma}$. The pressure scale height is $\frac{P}{g\rho}$. Therefore,

$$\begin{aligned} N^2 &= \frac{\gamma-1}{\gamma}\frac{g^2\rho}{P} && (23) \\ &= \frac{\gamma-1}{\gamma}\frac{g^2\mu m_u}{kT} && (24) \\ &= \frac{\gamma-1}{\gamma}\frac{G^2M^2\mu m_u}{R^4kT} && (25) \end{aligned}$$

Earth is about 80% N_2 and 20% O_2, so $\mu = 28.8$. These have three spacial and two rotational degrees of freedom, so $\gamma = 7/5$. Mars is mostly CO_2, so $\mu = 44$. This has three spacial and two rotational degrees of freedom. There are no vibrational modes at this teperature, so $\gamma = 4/3$ In addition, $T_\oplus = 288\text{K}$ and $T_{mars} = 210\text{K}$ on average. Plugging these values in yields:

$$\begin{aligned} N_{earth} &= 0.0182\,\text{s}^{-1} && (26) \\ N_{mars} &= 0.0093\,\text{s}^{-1} && (27) \end{aligned}$$

These correspond to periods of 5.75 and 11.3 minutes, respectively.

From problem 1, the acoustic wave cutoff frequency is

$$\omega_c^2 = \frac{1}{4}\gamma\frac{g^2\rho}{P} \tag{28}$$

We can substitute in the Brunt-Väisälä freqency to get

$$\omega_c^2 = \frac{1}{4}\frac{\gamma^2}{\gamma-1}N^2 \tag{29}$$

or

$$\omega_c = \frac{1}{2}\sqrt{\frac{\gamma^2}{\gamma-1}}N \tag{30}$$

For Earth, this gives

$$\omega_c = 0.0201\,\mathrm{s}^{-1} \tag{31}$$

which corresponds to a period of 5.20 minutes.

A wave in the frequency interval between ω_c and N is effanescent: it cannot propagate.

Problem 4:

4a:

We start by looking at hydrostatic equilibrium:

$$\frac{dP}{dR} = -g\rho \tag{32}$$

Because P changes very quickly near the surface, we cannot use an average value by approximating P as linear. Instead, we should put it into a form involving the sound speed, which more closely represents a linear relationship. Using our polytropic equation of state and sound speed equation, we get:

$$P \simeq \rho^{\gamma} \simeq (c^2)^{\gamma/(\gamma-1)} \tag{33}$$

which leads to

$$\frac{dP}{P} = \frac{\gamma}{\gamma-1}\frac{dc^2}{c^2} \tag{34}$$

This back into hydrostatic equilibrium gives

$$\frac{1}{c^2}\frac{dc^2}{dR} = -\frac{\gamma-1}{\gamma}g\frac{\rho}{P} \tag{35}$$

Plugging in $c^2 = \gamma\frac{P}{\rho}$ gives

$$\frac{dc^2}{dR} = -(\gamma-1)g \tag{36}$$

We now approximate the sounds speed as increasing linearly with depth to find

$$c^2 = (\gamma-1)gz \tag{37}$$

4b:

The distance that a p-wave can propagate into a star, given a fixed k_y and ω is found when $k_x = 0$ in our p-wave solution:

$$\omega^2 = c^2k_y^2 \tag{38}$$

Plugging in for c^2 and solving for $z_{\max}$ yields

$$\boxed{z_{\max} = \frac{\omega^2}{k_y^2(\gamma-1)g} = \frac{R^2\omega^2}{l^2(\gamma-1)g}} \tag{39}$$

4c:

The n, l, and ω relation for a standing wave is found by performing the following integral:

$$n\pi = \int_0^{z_{\max}} k_r dr \tag{40}$$

k_r is found by rearranging the p-wave relation and solving for k_x:

$$n\pi = \int_0^{z_{\max}} \sqrt{\frac{\omega^2}{(\gamma-1)gz} - k_r dz} \tag{41}$$

which becomes:

$$n\pi = \frac{\omega^2}{(\gamma-1)g}\frac{R}{l}\int_0^1 \sqrt{\frac{1}{x} - 1dx} \tag{42}$$

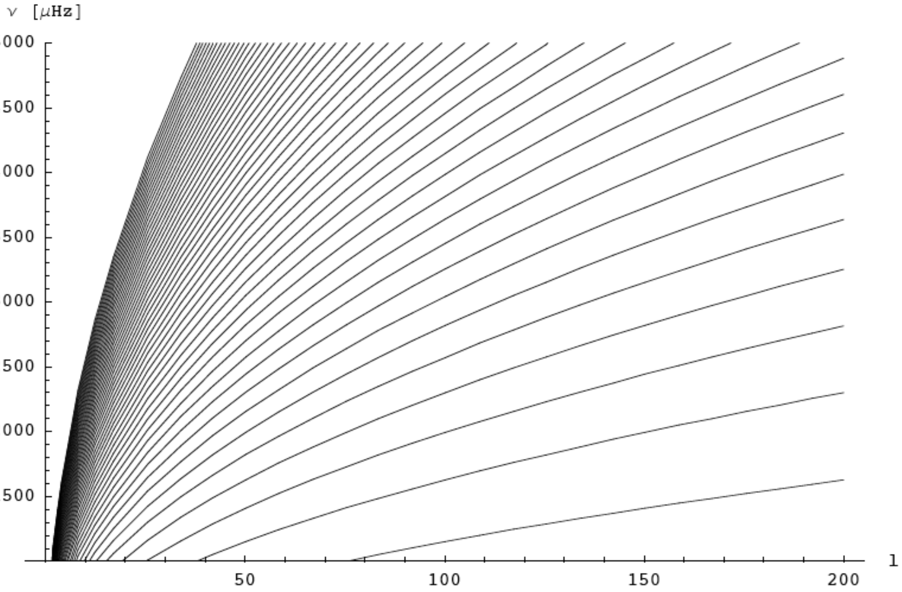
This can then be evaulated using an integral table to find:

$$\boxed{\omega^2 = 2n(\gamma-1)g\tfrac{l}{R}} \tag{43}$$

4d:

The above relation shows that the separation between lines of constant l should change as $\sqrt{n}$, as it seems to do in the figure of `astro-ph/0207403`. In addition, lines of constant n change as $\sqrt{l}$, as in the image. For a better comparison, let's recreate the same plot.

As can be seen, our graph compares favorably with the figure in the paper. However, our graph is slightly off by an overall scaling factor ($\simeq 10\%$). This could be due to our assumption that g is constant. If the average g value (from 0 to $z_{\max}$, weighted by the time it sepnds at that distance, which favors values close to $z_{\max}$) is less than the surface value, it would help bring the two plots into agreement. In addition, we may be off at very low values of l. This is because we should use $l(l+1)$ instead of l^2 and, as noted in the paper, $l + 1/2$ should be used instead of l at very small values.


ν [μHz]
5000
4500
4000
3500
3000
2500
2000
1500
50
100
150
200
l

Problem 5:

5a:

HKT equation (8.82) is

$$\rho' + \triangledown \cdot \left(\rho \vec{\xi}\right) = 0 \tag{44}$$

We assume that water is incompressible, so the Eulerian perturbation ρ' is zero and ρ is constant. This implies:

$$\rho \triangledown \cdot \vec{\xi} = 0 \tag{45}$$

which in turn implies:

$$\boxed{\triangledown \cdot \vec{\xi} = 0} \tag{46}$$

Now, HKT (8.80) is:

$$\rho \frac{\partial^2 \vec{\xi}}{\partial t^2} = - \triangledown P - \rho \triangledown \Phi - \triangledown P' - \rho \triangledown \Phi' - \rho' \triangledown \Phi \tag{47}$$

Now, we assume that gravity is constant, so $\triangledown \Phi' = 0$. Also, hydrostatic equilibrium guarantees that $\triangledown P = -\rho \triangledown \Phi$. The above then becomes:

$$\rho \frac{\partial^2 \vec{\xi}}{\partial t^2} = - \triangledown P' \tag{48}$$

Taking the divergence and making using use of incompressibility then implies

$$\boxed{\triangledown^2 P' = 0} \tag{49}$$

5b:

Rewriting the above Laplacian equation:

$$\frac{\partial^2 P'}{\partial z^2} = - \frac{\partial^2 P'}{\partial x^2} \tag{50}$$

We now assume a solution of the form:

$$P' = w(z) \cos(kx) \tag{51}$$

to find:

$$\frac{d^2w}{dz^2}\cos(kx) = -w(z)\left(-k^2\cos(kx)\right) \tag{52}$$

and so:

$$\boxed{\tfrac{d^2w}{dz^2} = k^2 w} \tag{53}$$

The general solution to this equation is

$$w(z) = A\exp(-kz) + B\exp(kz) \tag{54}$$

We disregard the divergent solution to find:

$$\boxed{w(z) \propto \exp(-kz)} \tag{55}$$

5c:

In analogy with HKT (8.79), we have:

$$\delta P = P' + \vec{\xi}\cdot\bigtriangledown P \tag{56}$$

At the surface, δP must vanish. Using this, hydrostatic equilibrium ($dP/dz = g\rho$), and the result of the previous question:

$$0 = \exp(-kz)\cos(kx) + \xi_z g\rho \tag{57}$$

near the surface.

Next, we return to (8.80) which will, we recall, only keep the terms

$$\bigtriangledown P' = -\rho\frac{\partial^2\vec{\xi}}{\partial t^2} \tag{58}$$

Evaluating the derivatives while assuming a wave solution for $\vec{\xi}$ gives

$$-k\exp(-kz)\cos(kx) = \sigma^2\rho\xi_z \tag{59}$$

for the z-component.

We now plug in our earlier equation for P' to find:

$$k(\xi_z g\rho) = \sigma^2\rho\xi_z \tag{60}$$

or

$$\boxed{\sigma^2 = gk} \tag{61}$$

5d:

From the above dispersion relation, the period as a function of wavelength is

$$\frac{4\pi^2}{T^2} = g\frac{2\pi}{\lambda} \tag{62}$$

or

$$T = sqrt\frac{2\pi\lambda}{g} \tag{63}$$

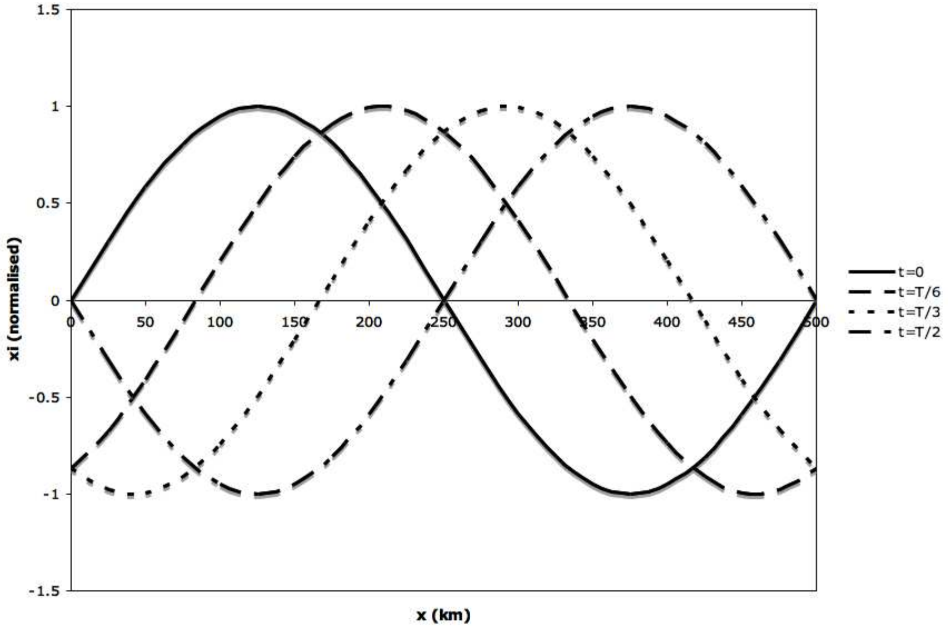
For a $\lambda = 500\,\mathrm{km}$ tsunami, this gives

$$\boxed{T = 9.4\,\mathrm{min}} \tag{64}$$

The speed of such a wave is

$$\boxed{v = \tfrac{\lambda}{T} = 3200\,\mathrm{km/h}} \tag{65}$$

See the sketches below.


xi (normalised)
1.5
1
0.5
0
-0.5
-1
-1.5
0
50
100
150
200
250
300
350
400
450
500
x (km)
t=0
t=T/6
t=T/3
t=T/2