# Solution: HW 6
# AY 123, Fall 2007

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## Problem 1:

### 1a:

We are considering a first order ordinary differential equation of the form:

$$\frac{dy}{dx} = f(x) \tag{1}$$

We are interested in solving for $y(x)$. We will evaluate the overall error of a method by considering the error accumulated in a single step of size $h$.

The Taylor expansion of $y(x)$ around some point $x_0$ is:

$$y(x_0 + h) = y(x_0) + hf(x_0) + \frac{1}{2}h^2 f'(x_0) + \frac{1}{6}h^3 f''(x_0) + \ldots \tag{2}$$

The Euler scheme for numerical evaluation of the differential equation, however, makes the approximation:

$$y(x_0 + h) \simeq y(x_0) + hf(x_0) \tag{3}$$

Clearly, the difference between the exact and approximate values for $y(x_0 + h)$ scales as $h^2$. Hence we have

$$\boxed{\epsilon \propto h^2} \tag{4}$$

**1b:**

Using a second order Runge-Kutta integration, the step will take the form:

$$y(x_0 + h) \simeq y(x_0) + hf(x_0 + h/2) \tag{5}$$

The second term can be Taylor expanded:

$$f(x_0 + h/2) = f(x_0) + \frac{h}{2}f'(x_0) + \frac{h^2}{8}f''(x_0) + \dots \tag{6}$$

Inserting this back into the Runge-Kutta expression gives:

$$y(x_0 + h) \simeq y(x_0) + hf(x_0) + \frac{h^2}{2}f' + \frac{h^3}{8}f''(x_0) + \dots \tag{7}$$

Comparison with the correct Taylor expansion shows an error of order:

$$\boxed{\epsilon \propto h^3} \tag{8}$$

**1c:**

The differential equation

$$\frac{dy}{dx} = x^2 + \sin(x) \tag{9}$$

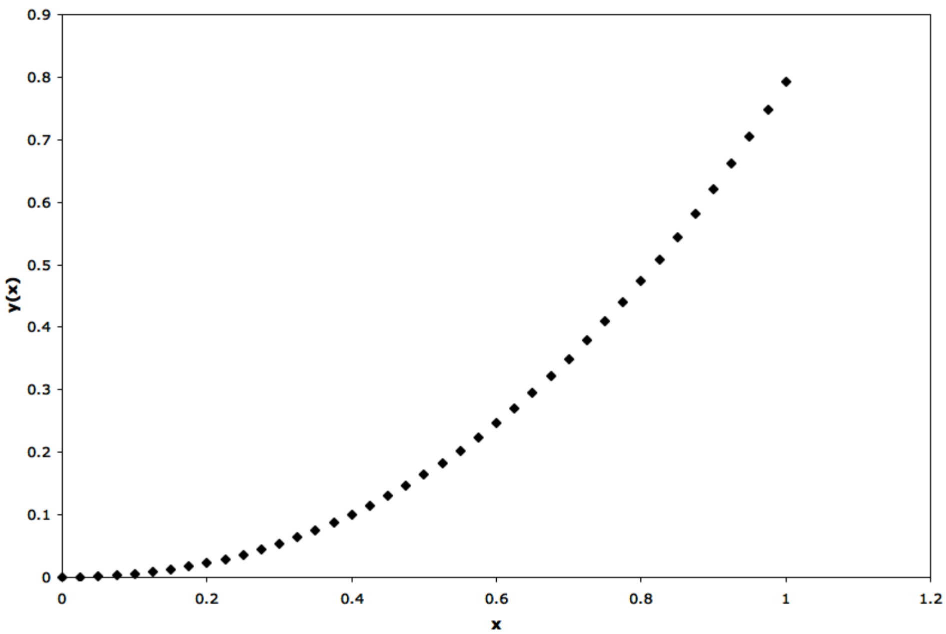
is trivially integrated with $y(0) = 0$ to get

$$y(x) = \frac{1}{3}x^3 - \cos(x) + 1 \tag{10}$$

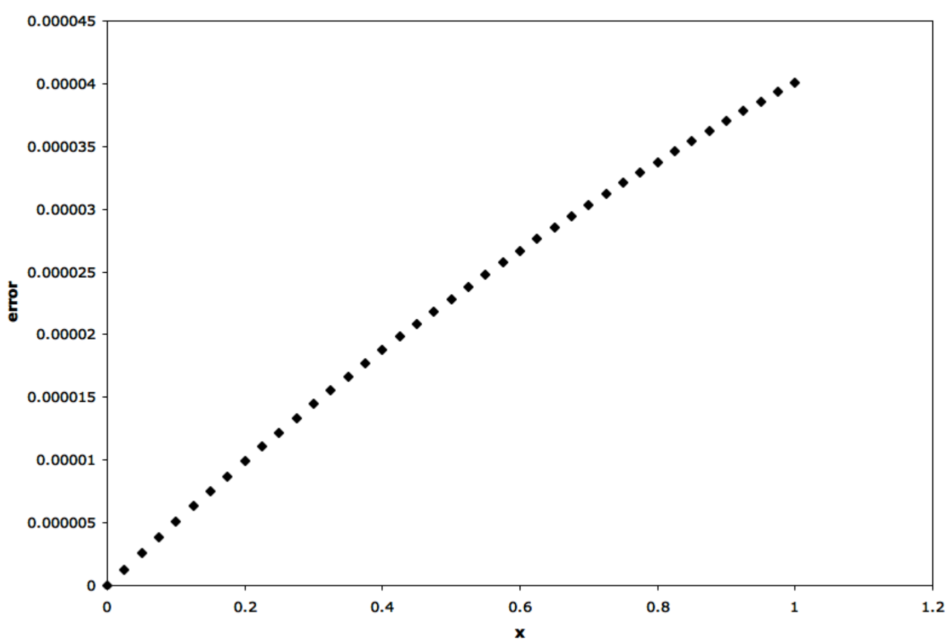
A plot is included.

**1d:**

Using a Runge-Kutta scheme, the above differential equation is evaluated. The resulting plot is indistinguishable from the analytic solution. The minute difference is shown in a difference plot.



y(x)
x

error
x

## Problem 2:

### 2a:

**Unless listed otherwise, all quantities are in Solar units. Time is in units of Myr.**

We begin by listing the tools at our disposal. First, we have an initial mass-radius relation:

$$R = 54.88M \tag{11}$$

for $t = 0$. Next, we have the $L - T - M$ relation for the Hayashi track:

$$\log(L) = 15\log(T) + 0.2\log(M) + \delta \tag{12}$$

where $\delta$ is to be solved for assuming at $t = 0$ an $M = 1$ star has $L = 40$. This can be done when we recall

$$L = T^4 R^2 \tag{13}$$

Combining the above three equations yields:

$$\delta = 8.64 \tag{14}$$

Next, we have the Henyey track relation:

$$L = AM^{5.5}R^{-0.5} \tag{15}$$

where $A$ is mass dependent. We also have the virial luminosity relation

$$L = \frac{1}{2}\frac{d\Omega}{dt} = 6.34\frac{M^2}{R^2}\frac{dR}{dt} \tag{16}$$

Finally, we are given the main sequence relations

$$L = M^3 \tag{17}$$

and

$$R \propto M^{(n-1)/(n+3)} \tag{18}$$

where $n = 4$ for $M < 2$ and $n = 16$ for $M > 2$. This difference will result in a difference in proportionality constant. Matching the relations at $M = 2$ gives

$$R = M^{3/7} \tag{19}$$

$$R = 0.8524M^{15/19} \tag{20}$$

for low and high mass, respectively.

The next step is to find the intersection points of the Hayashi and Henyey tracks for each mass which lead to the correct main sequence positions. We can ensure the correct main sequence positions by solving for $A$ for each mass. We do this by plugging the main sequence equations for $L$ and $R$ to find

$$A = M^{-2.286} \tag{21}$$
$$A = 0.9233M^{-2.105} \tag{22}$$

for low and high mass, respectively.

Next, we eliminate $L$ and $T$ between the Hayashi track equation, the Henyey track equation, and the $L-T-R$ equation. This results in a relation between $M$ and $R$ at the switching point:

$$\log R = 1.727 \log M + \frac{2.75 \log A + \delta}{8.875} \tag{23}$$

This radius will then be plugged into the Henyey equation to find $L$, and in turn into the $L - T - R$ relation to find $T$. The values for the masses are compiled in the table below:

| Mass | Radius | Luminosity | Temp (K) |
|---|---|---|---|
| 0.5 | 4.64 | 0.0500 | 1270 |
| 1 | 9.41 | 0.3260 | 1420 |
| 2 | 19.06 | 2.125 | 1600 |
| 4 | 40.72 | 16.01 | 1810 |
| 8 | 85.79 | 116.07 | 2050 |
| 16 | 180.68 | 841.25 | 2320 |

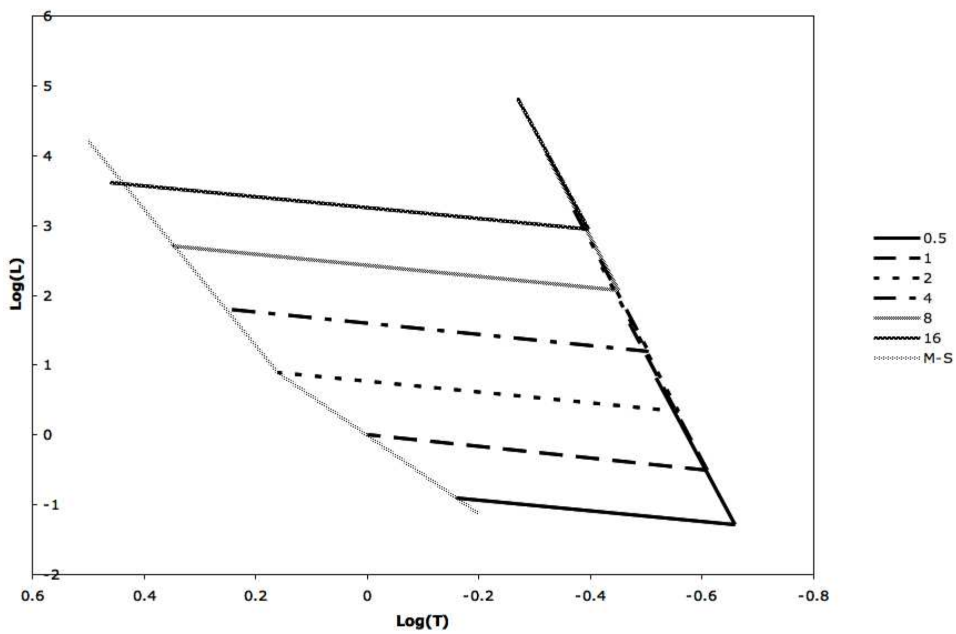
The evolutionary tracks are plotted below.

**2b:**

We are now looking for the times of the tracks. We solve this by through use of the luminosity differential equation.

First for the Hayashi track:

$$-2.75 \log L = -7.5 \log R + 0.2 \log M + \delta \tag{24}$$
$$L^{-2.75} = 10^{\delta} R^{-7.5} M^{0.2} \tag{25}$$




Eliminating $L$ between this and the differential equation:

$$6.34\frac{M^2}{R^2}\frac{dR}{dT} = 10^{-\delta/2.75}R^{30/11}M^{-8/110} \tag{26}$$

which when sorted out becomes:

$$\frac{dR}{dt} = 1.138 \times 10^{-4}R^{4.727}M^{-2.073} \tag{27}$$

This is integrated to give:

$$t_1 = 2358M^{2.073}\left(\frac{1}{R_1^{3.727}} - \frac{1}{R_0^{3.727}}\right) \tag{28}$$

The amount of time spent on the Henyey track is done in the same way, but with the Henyey relation used in place of the Hayashi relation:

$$6.34\frac{M^2}{R^2}\frac{dR}{dT} = AM^{5.5}R^{-0.5} \tag{29}$$

which becomes:

$$\frac{dR}{dt} = \frac{A}{6.34}M^{3.5}R^{1.5} \tag{30}$$

This is integrated to give:

$$t_2 - t_1 = \frac{12.68}{A} M^{-3.5} \left( \frac{1}{R_2^{0.5}} - \frac{1}{R_1^{0.5}} \right) \tag{31}$$

Plugging in the relevant values, we find the following. Recall that the times are in units of Myr.

| Mass | $t_1$ | $t_2 - t_1$ | $t_2$ |
|---|---|---|---|
| 0.5 | 1.83577 | 20.46969 | 22.30547 |
| 1 | 0.55382 | 8.54644 | 9.10026 |
| 2 | 0.16785 | 3.46008 | 3.62793 |
| 4 | 0.04169 | 0.93301 | 0.97470 |
| 8 | 0.01091 | 0.27827 | 0.28918 |
| 16 | 0.00286 | 0.08272 | 0.08558 |