Solution: HW 7
AY 123, Fall 2007

by Matthew Stevenson

December 5, 2007

Problem 1:

We will make the following assumptions:

- a plane-parallel atmosphere,
- constant surface gravity,
- hydrostatic and radiative equilibrium,
- local thermodynamic equilibrium,
- an ideal equation of state,
- a grey atmosphere, including the Eddington approximation,
- opacity due to the H^- ion at 500 nm, and
- the helium remained neutral and metals were limited to single ionization.

The subroutines are described below. The ratio (by number) of helium to hydrogen is B and for metals to hydrogen is A. The gravity g and effective temperature are parameters to the model.

- **Hydrostatic Equilibrium**
  Inputs: κ; Outputs: P We are using pressure P over optical depth τ as our primary function. We can integrate our differential equation numerically:
  $$\frac{dP}{d\tau} = \frac{g}{\kappa} \tag{1}$$
  where κ is the opacity.

- **Runge-Kutta**
  Inputs: P, $\Delta\tau$; Outputs $P+\Delta P$ We integrate inwards using the Runge-Kutta method. A 2nd order scheme will take the form:
  $$P_{n+1} \simeq P_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2}$$
  $$k_1 = \Delta\tau f(P_n, \tau_n) \tag{3}$$
  $$k_2 = \Delta\tau f(P_n + k_1/2, \tau_n + h/2) \tag{4}$$
  $$k_3 = \Delta\tau f(P_n + k_2/2, \tau_n + h/2) \tag{5}$$
  $$k_4 = \Delta\tau f(P_n + k_3, \tau_n + h) \tag{6}$$
  where
  $$P' = f(P, \tau) \tag{7}$$

- **Temperature**
  Input: τ; Output: T By assuming a grey atmosphere with the Eddington approximation, we have the temperature specified as a function of optical depth:
  $$T^4 = \frac{3}{4}T_{\text{eff}}^4(\tau + 2/3) \tag{8}$$

- **Saha Equation**
  Inputs: T, P_e; Outputs: x, y The Saha equation is used to calculate the ionization fractions of hydrogen and metals, x and y.
  $$\frac{n_1}{n_0} = \frac{2}{P_e}\frac{u_1}{u_2}\frac{(2\pi m)^{3/2}}{h^3}(kT)^{5/2}\exp(-\chi_0/kT) \tag{9}$$
  This depends on the electron pressure P_e which comes from the equation of state. The equation of state depends on x and y, so these two equations must be solved in parallel.

- **Equation of State** Inputs: T, P, x, y; Output: P_e We assume an ideal gas:

$$P = nkT \tag{10}$$

where

$$n_e = (x + y/A)n_H \tag{11}$$

$$n_{\text{nuc}} = (1 + 1/A + B)n_H \tag{12}$$

n_H is found from the total pressure and in turn used to find P_e. This must be solved in tandem with the Saha equation.

- **Secant Method**
Inputs: P, T; Outputs: x, y, P_e This is a numerical method for solving parallel algebraic equations. It is a variant of Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{13}$$

when we don't know the derivative:

$$f' \simeq \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \tag{14}$$

This can be applied to the Saha equation and the Equation of State to solve for x, y, and P_e.

- **Opacity**
We assume opacity due to H^- opacity:

$$\kappa = \frac{(1 - x)}{(1 + 4B)m_H}\alpha \tag{15}$$

where the complicated function $\alpha(T, P_e, \lambda = 500\,\text{nm})$ is given in `Mihalas, D. 1967, in Methods of Computational Physics Vol 7, ed. Alder, Fernbach, & Rotenberg (New York: Academic Press).`

- **Density**
Inputs: x, y, P, T; Output: ρ The density is a quantity of interest. It can be calculated from the ideal gas equation:

$$\rho = \frac{\mu P}{N_A kT} \tag{16}$$

- **Physical Depth**
  Inputs: κ, ρ, z; Output: $z + \Delta z$ We would like the physical depth as a function of optical depth. We can find this from the definition of optical depth:
  $$dz = \frac{d\tau}{\kappa\rho} \tag{17}$$

- **Boundary Conditions**
  Inputs: κ, τ_{surf}; Output: P_{surf} We need to begin the integration by taking some condition at the outer boundary. We do this by linearizing the hydrostatic equilibrium equation near the surface, which is taken as some small τ_{surf}. Note that κ requires T and P_{surf}, so the equation is best solved using the secant method:
  $$P_{\mathrm{surf}} = \frac{g\tau_{\mathrm{surf}}}{\kappa(P_{\mathrm{surf}}, \tau_{\mathrm{surf}})} \tag{18}$$

- **Output**
  Inputs: All variables As we integrate inwards, we will want to write all of the calculated variables to file. These form our model atmosphere for use with further work.

An effective flowchart is included.

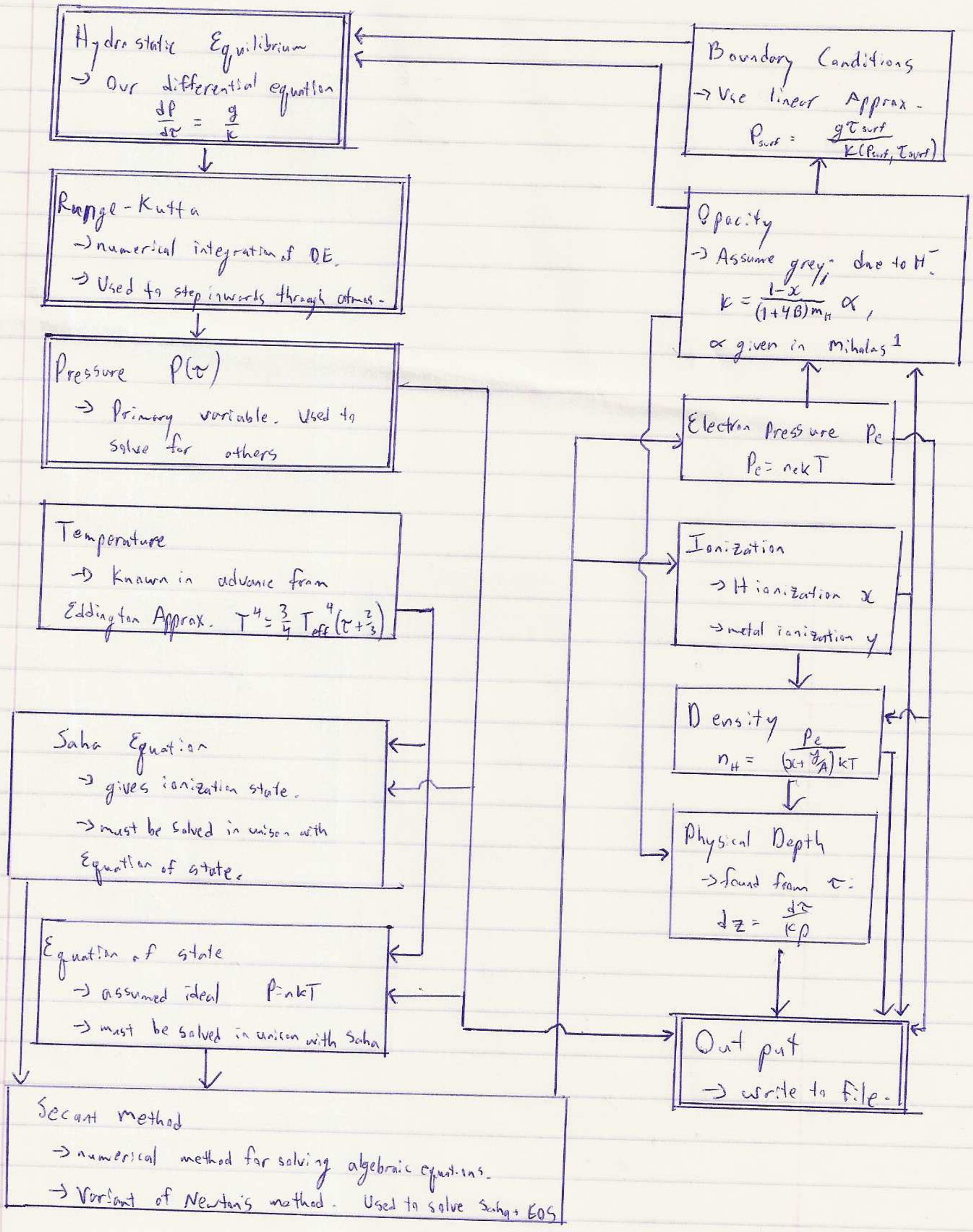
We may also choose to calculate the continuum spectrum of the star at the centre of its disk. We can do so using the solution to the equation of transfer:

$$I_\lambda = \int_0^\infty S_\lambda \mathrm{e}^{-\tau'} d\tau' \tag{19}$$

where S_λ is the source function from blackbody radiation. We know $T(\tau)$, so we can calculate this directly. It can be done numerically using the roots of Laguerre polynomials:

$$\int_0^\infty \mathrm{e}^{-x} f(x)dx \simeq \sum_{i=1}^{n} w_i f(x_i) \tag{20}$$

The weights w_i are available, for instance, at `http://www.nr.com`.



[1] Mihalas, D. 1967, in Methods of Computational Physics Vol. 7, ed. Alder, Fernbach, Rotenberg (New York: Academic Pr.)

Problem 2:

2a:

The escape speed is found by equating kinetic energy to gravitational:

$$\frac{GMm}{R} = \frac{1}{2}mv^2 \tag{21}$$

which gives

$$v = \sqrt{\frac{2GM}{R}} \tag{22}$$

For a $5\,\mathrm{M}_\odot$, $20\,\mathrm{R}_\odot$ star, this gives

$$\boxed{v = 3.1 \times 10^7\,\mathrm{cm/s}} \tag{23}$$

2b:

If the star is losing mass at a rate $\dot{M}$ and speed v, then the mass per unit radius is simply:

$$\lambda(r) = \frac{\dot{M}}{v} \tag{24}$$

The space density as a function of radius is then:

$$\rho(r) = \frac{\dot{M}}{4\pi r^2 v} \tag{25}$$

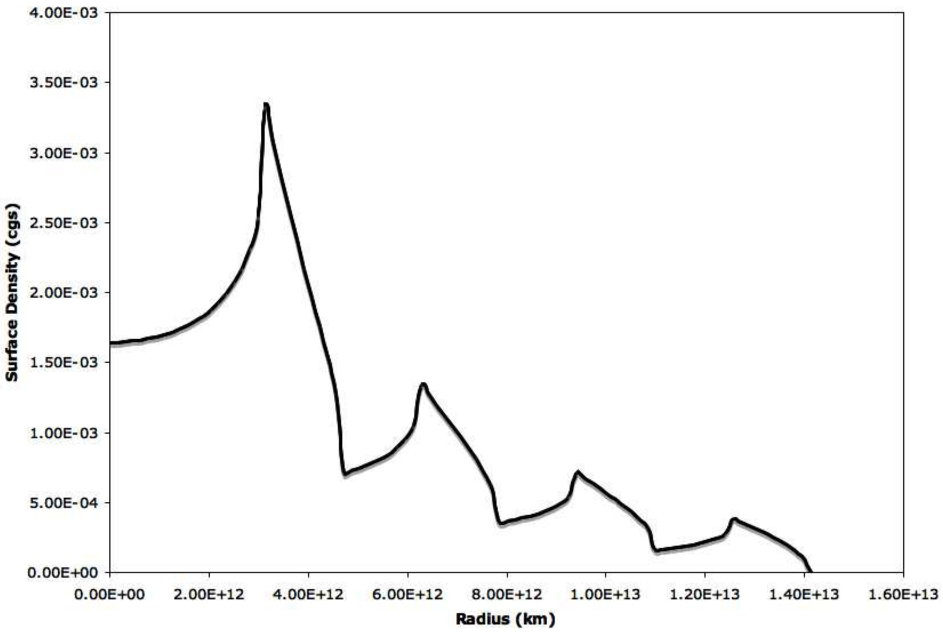
where, for a variable $\dot{M}$, the above $\dot{M}$ will be the mass-loss rate at time $t_0 = t - r/v$.

We can then calculate the surface density as seen on the celestial sphere via:

$$\sigma(R) = \int_{-\infty}^{\infty} \rho(R^2 + z^2)dz \tag{26}$$

In order to evaluate this for the piecewise nature of this problem, it will be evaluated for a single ring of arbitrary $\dot{M}$, inner radius r_1, and outer radius r_2:

$$\sigma_i(R) = \frac{\dot{M}}{2\pi R v}\left(\arccos(R/r_2) - \arccos(R/r_1)\right) \tag{27}$$



Using this for each of the eight shells, we can then calculate the requested surface density profile. See the attached figure.

The total mass of the star after the 4 episodes of mass loss is

$$M = 5\,\mathrm{M}_\odot - 4\,5000\left(10^{-4} + 10^{-8}\right) \tag{28}$$

or

$$\boxed{M = 2.9998\,\mathrm{M}_\odot} \tag{29}$$

2c:

Let us now assume that the ionizing radiation is strong enough to completely ionize the halo. Let us also assume that our emission line is optically thin. Finally, let us assume that every recombination results in the emission of photon of interest.

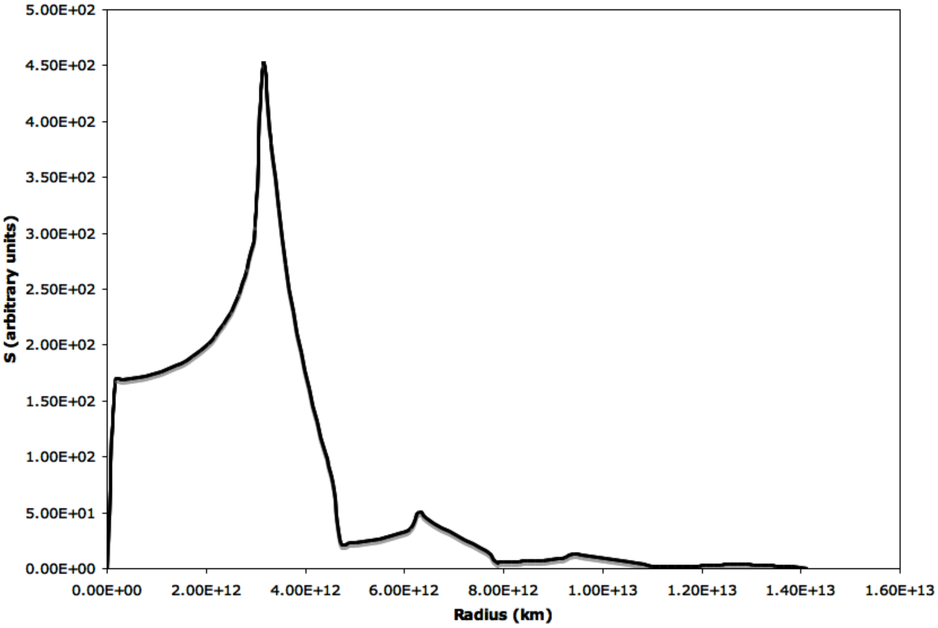
The strength of the emission line will be proportional to the number of recombinations per second. The number of recombinations per second goes as the square of the density, so we have

$$S_\nu(R) \propto \int_{-\infty}^{\infty} \rho^2 dz \tag{30}$$

Again integrating this for just one of the mass-loss shells:

$$S_{\nu,i}(R) \propto \frac{\dot{M}^2}{R^3}\left(0.5\sin(2\arccos R/r_2) + \arccos(R/r_2) - 0.5\sin(2\arccos R/r_1) - \arccos(R/r_1)\right) \tag{31}$$

This is plotted for an arbitrary proportionality constant.



Problem 3:

3a:

We begin with the white dwarf relation

$$T_c = (7 \times 10^7\,\mathrm{K}) \left(\frac{L/L_\odot}{M/M_\odot} \right)^{2/7} \tag{32}$$

Further,

$$\frac{dE}{dt} = -L \tag{33}$$

where

$$E = \frac{3}{2} \frac{M}{12 m_H} k T_c \tag{34}$$

Combining these,

$$\frac{dT_c}{dt} = -\frac{2}{3} \frac{12 m_H}{M_\odot} \frac{L_\odot}{k(7 \times 10^7\,\mathrm{K})} T_c^{7/2} \tag{35}$$

This is trivial to integrate. Doing so and plugging in the constants along with $T_{c,i} = 10^7\,\mathrm{K}$, we find

$$\boxed{T_c = \left(5.1 \times 10^{-27} t\,[\mathrm{yrs}] + 3.16 \times 10^{-18}\right)^{-2/5}\ \mathrm{K}} \tag{36}$$

Now, the mass-radius relation for white dwarfs is

$$\frac{M}{M_\odot} \simeq 10^{-6} \left(\frac{R}{R_\odot} \right)^{-3} \left(\frac{2}{\mu_e} \right)^5 \tag{37}$$

where, in this case, $\mu_e = 2$. We then find

$$\frac{R}{R_\odot} = 10^{-2} \left(\frac{M}{M_\odot} \right)^{-1/3} \tag{38}$$

Combining this with the definition of effective temperature:

$$\frac{L}{L_\odot} = \left(\frac{T_{\mathrm{eff}}}{T_{\mathrm{eff},\odot}} \right)^4 \left(\frac{R}{R_\odot} \right)^2 \tag{39}$$

and eliminating the luminosity with equation 32:

$$T_{\text{eff}} = 10 \left[\left(\frac{T_c}{7 \times 10^7\,\text{K}} \right)^{7/2} \left(\frac{M}{M_\odot} \right)^{5/3} \right]^{1/4} T_{\text{eff},\odot} \tag{40}$$

This is combined with the result for T_c and applied to a $0.6\,\text{M}_\odot$ white dwarf:

$$\boxed{T_{\text{eff}} = 0.00637\,(5.1 \times 10^{-27} t\,[\text{yrs}] + 3.16 \times 10^{-18})^{-0.35}\ \text{K}} \tag{41}$$

3b:

The Salpeter initial mass function is

$$n(M) = AM^{-(1+x)} \tag{42}$$

Given a total of 10^6 stars between 0.1 and $100\,\text{M}_\odot$, A can be evaluated. Using $x = 1.35$:

$$10^6 = A \int_{0.1}^{100} M^{-2.35} dM \tag{43}$$

$$A = 6.03 \times 10^4 \tag{44}$$

We find the number of Type II supernovae by integrating the IMF from 8 to $100\,\text{M}_\odot$:

$$N_{\text{SN}} = 6.03 \times 10^4 \int_{8}^{100} M^{-2.35} dM \tag{45}$$

to find:

$$\boxed{N_{\text{SN}} = 2600} \tag{46}$$

The number of white dwarfs is found by performing the same integral, except over the range 1 to $8\,\text{M}_\odot$. The result is:

$$\boxed{N_{\text{WD}} = 4.2 \times 10^4} \tag{47}$$

We find the number of stars still on the main sequence by integrating once more, this time between 0.1 and $1\,\text{M}_\odot$. The result is:

$$\boxed{N_{\text{MS}} = 9.6 \times 10^5} \tag{48}$$

The total mass in a given mass range is found by the integral:

$$\Delta M = \int_{M_{\rm min}}^{M_{\rm max}} M' n(M') dM' \tag{49}$$

The total mass originally on the main sequence is found by integrating the above from 0.1 to $100\,\mathrm{M}_\odot$. This rather trivial integral yields:

$$\Delta M_{\rm tot} = 3.5 \times 10^5\,\mathrm{M}_\odot \tag{50}$$

The total mass still on the main sequence is found by integrating from 0.1 to $1\,\mathrm{M}_\odot$, which yields:

$$\Delta M_{\rm MS} = 2.1 \times 10^5\,\mathrm{M}_\odot \tag{51}$$

The fraction of mass which is still on the main sequence is merely the ratio of these two values:

$$\boxed{f_{\rm MS} = 0.6} \tag{52}$$

We now want to calculate a luminosity function for the white dwarfs. Assuming white dwarfs have the same properties as in 3a, the luminosity of a given white dwarf will be a function of the time spent as a white dwarf (or, in this problem, time since leaving the main sequence). We must therefore find the amount of time that a given star will spend on the main sequence.

For stars of order a few solar masses:

$$L \propto M^{3.5} \tag{53}$$

so

$$t_{\rm MS} \propto \frac{M}{L} \propto M^{-2.5} \tag{54}$$

Normalizing to the solar values:

$$t_{\rm MS} = 10 \left(\frac{M}{M_\odot}\right)^{-2.5} \mathrm{Gyr} \tag{55}$$

Now, returning to the $T_{\rm eff}$ relation from the previous section and calculating from it the luminosity (also using the $M - R$ relationship and a 0.6 solar mass white dwarf):

$$\frac{L}{L_\odot} = 2.1 \times 10^{-28} \left[5.1 \times 10^{-27}(t\,[\mathrm{yrs}] - t_{\rm MS}\,[\mathrm{yrs}]) + 3.16 \times 10^{-18}\right]^{-1.4} \tag{56}$$

Replacing t_{MS} with the mass relation from above, setting $t = 10\,\mathrm{Gyr}$, and solving for M:

$$\frac{M}{M_\odot} = 3.0 \times 10^{-7} \left[5.4 \times 10^{-17} - 1.7 \times 10^{-20} \left(\frac{L}{L_\odot} \right)^{-5/7} \right]^{-2/5} \tag{57}$$

Now, the luminosity function is found by

$$n(L) = n(M) \left| \frac{dM}{dL} \right| \tag{58}$$

Differentiating the above $M - L$ relationship:

$$\frac{dM/M_\odot}{dL/L_\odot} = -1.4{\times}10^{-27} \left[5.4 \times 10^{-17} - 1.7 \times 10^{-20} \left(\frac{L}{L_\odot} \right)^{-5/7} \right]^{-7/5} \left(\frac{L}{L_\odot} \right)^{-12/7} \tag{59}$$

Pluggin this and the above $M - L$ relationship into the luminosity function formula:

$$n(L) = 5.5 \left(1 - 3.1 \times 10^{-4} L^{-5/7} \right)^{-0.46} L^{-1.71} \tag{60}$$

The exponent can be linearly expanded to give:

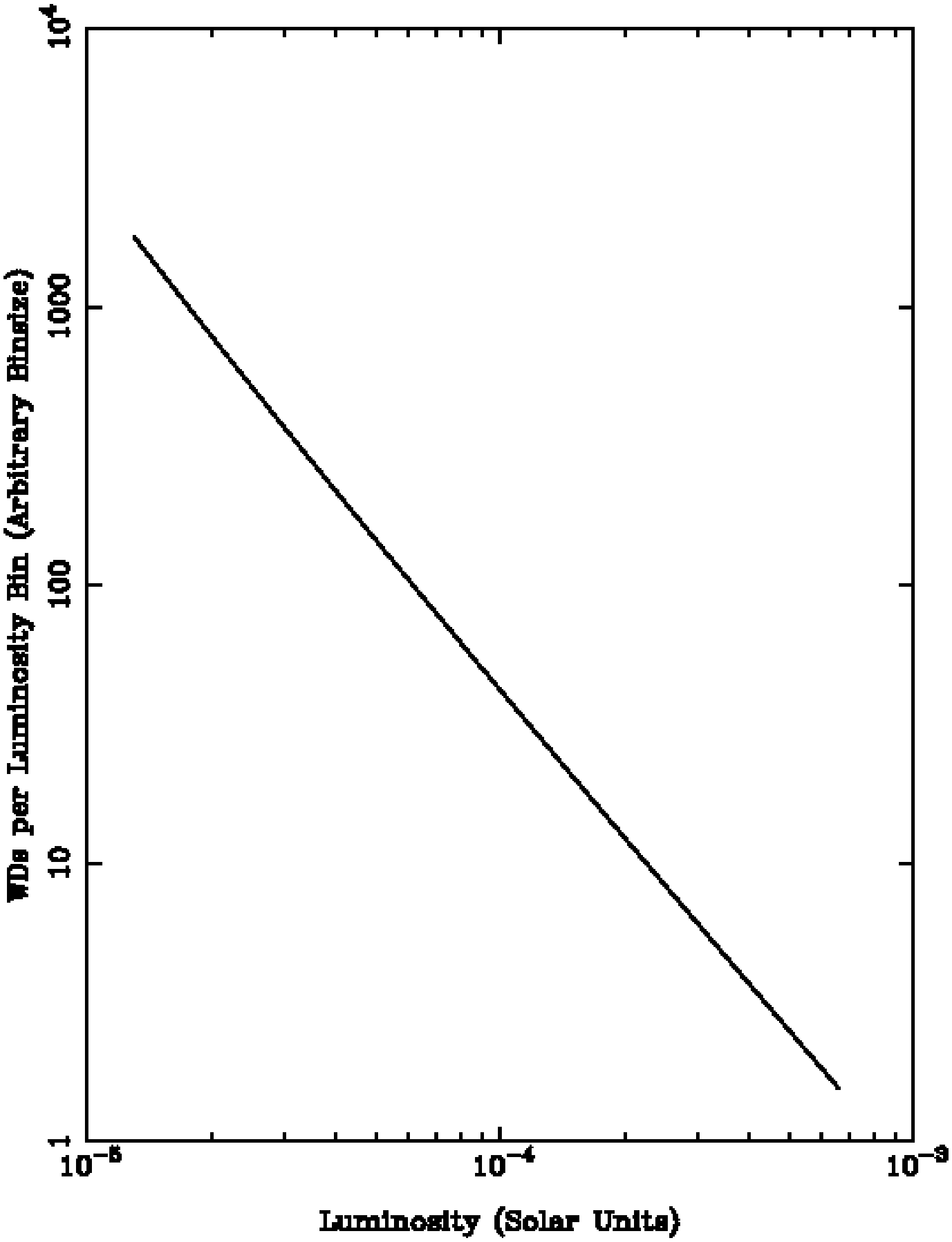
$$n(L) = 5.5 \left(1 + 1.4 \times 10^{-4} L^{-5/7} \right) L^{-1.71} \tag{61}$$

This function is only valid for initial masses between 1 and $8\,\mathrm{M}_\odot$, which translates to the range:

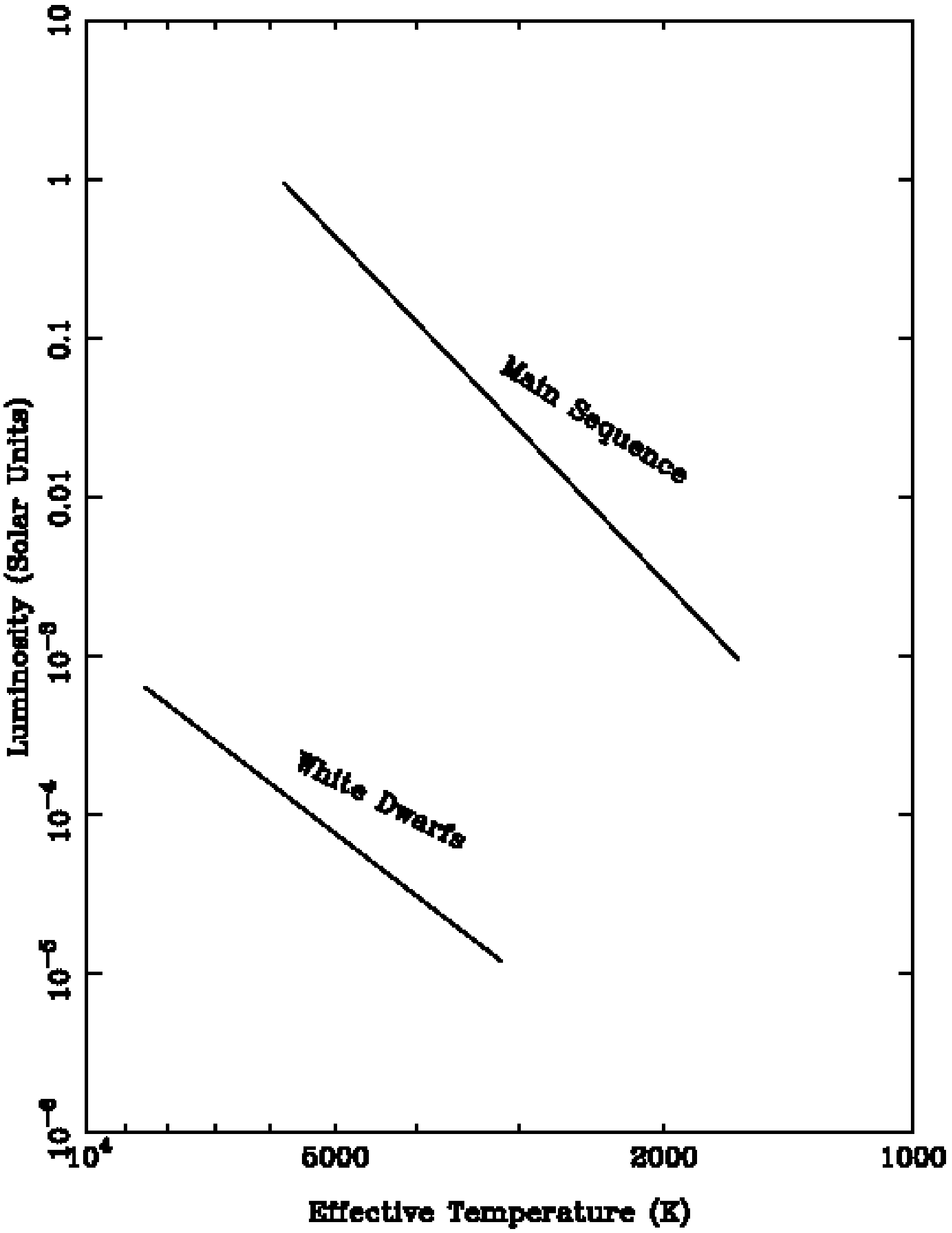
$$\boxed{n(L) = \begin{cases} 5.5 \left(1 + 1.4 \times 10^{-4} L^{-5/7} \right) L^{-1.71} & 1.3 \times 10^{-5} < L < 6.6 \times 10^{-4} \\ 0 & \text{otherwise} \end{cases}} \tag{62}$$

This is plotted.

As for the HR diagram, the white dwarf locus was plotted by evaluating the expressions for $T_{\mathrm{eff}}(t)$ and $L(t)$ derived earlier. The main sequence was plotted by assuming $T_{\mathrm{eff}} \propto M^{0.55}$ and $L \propto M^3$.



10^4
1000
100
10
1
10^{-5}
10^{-4}
10^{-3}
WDs per Luminosity Bin (Arbitrary Binsize)
Luminosity (Solar Units)


Luminosity (Solar Units)
10
1
0.1
0.01
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^4
5000
2000
1000
Effective Temperature (K)
Main Sequence
White Dwarfs