Solution: HW 8
AY 123, Fall 2007

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Problem 1:

The formula used below are taken from the chapter on opacities in Gray. We will consider H bound-free, H free-free, and Thomson scattering. Throughout the problem we approximate the gaunt factors to be unity.

The H bound-free absorption coefficient is given by:

$$\kappa_{\rm bf}(H) = \alpha_0 \frac{\lambda^3}{n_l^3} 10^{-\theta\chi_n}\,{\rm cm}^2/{\rm H} \tag{1}$$

where n_l is the absorbing excitation level, $\theta = 5040/T$, χ_n is the excitation energy of n_l, and

$$\alpha_0 = 1.04 \times 10^{-26} \tag{2}$$

and λ is in angstroms. This must be multiplied by the number of number density of neutral H atoms, given by the Saha equation (assuming mostly ionized):

$$n_H = n^2 \left(\frac{h^2}{2\pi m_e kT}\right)^{3/2} \exp\left(\frac{\chi_H}{kT}\right) \tag{3}$$

and the physical depth to give:

$$\tau(H_{\rm bf}; 500\,{\rm nm}) = 6.7 \times 10^{-6} \tag{4}$$

$$\tau(H_{\rm bf}; 100\,{\rm nm}) = 7.8 \times 10^{-7} \tag{5}$$

in the chromosphere; and

$$\tau(H_{\rm bf}; 500\,{\rm nm}) = 9.8 \times 10^{-16} \tag{6}$$
$$\tau(H_{\rm bf}; 100\,{\rm nm}) = 3.8 \times 10^{-16} \tag{7}$$

for the corona.

H free-free emission has a coefficient

$$\kappa_{\rm ff} = \alpha_0 \lambda^3 \frac{\log({\rm e})}{2\theta I} 10^{-\theta I} \tag{8}$$

where $\theta = 5040/T$ and $I = 13.6$. This is to be multiplied by neutral H number density and thickness to find:

$$\tau(H_{\rm ff}; 500\,{\rm nm}) = 2.6 \times 10^{-6} \tag{9}$$
$$\tau(H_{\rm ff}; 100\,{\rm nm}) = 1.8 \times 10^{-8} \tag{10}$$

in the chromosphere; and

$$\tau(H_{\rm ff}; 500\,{\rm nm}) = 1.3 \times 10^{-12} \tag{11}$$
$$\tau(H_{\rm ff}; 100\,{\rm nm}) = 1.0 \times 10^{-14} \tag{12}$$

for the corona.

Finally, for Thomson scattering, we have

$$\tau = \alpha_T n_e z \tag{13}$$

where $\alpha_T = 0.6648 \times 10^{-24}\,{\rm cm}^2/{\rm e}^-$. This is clearly independent of wavelength, so we find:

$$\tau = 1.3 \times 10^{-4} \tag{14}$$

for the chromosphere and

$$\tau = 4.6 \times 10^{-6} \tag{15}$$

for the corona.

Adding these up gives

$$\tau({\rm chromosphere}, 500\,{\rm nm}) = 1.4 \times 10^{-4} \tag{16}$$
$$\tau({\rm chromosphere}, 100\,{\rm nm}) = 1.3 \times 10^{-4} \tag{17}$$
$$\tau({\rm corona}, 500\,{\rm nm}) = 4.6 \times 10^{-6} \tag{18}$$
$$\tau({\rm corona}, 100\,{\rm nm}) = 4.6 \times 10^{-6} \tag{19}$$

Problem 2:

2a:

The equation of transfer

$$\mu \frac{dI_\nu}{d\tau_n u} = I_\nu - S_\nu \tag{20}$$

can be integrated over a semi-infinite plane-parallel body to find

$$I_\nu(\mu, \tau_\nu = 0) = \frac{1}{\mu} \int_0^\infty S_\nu \mathrm{e}^{-\tau_\nu/\mu} d\tau_\nu \tag{21}$$

We assume that

$$S_\nu = C_\nu + D_\nu \tau_\nu \tag{22}$$

Plugging this into the above integral gives

$$I_\nu(\mu, \tau_\nu = 0) = C_\nu + D_\nu \mu \tag{23}$$

Taking the ratio of this at arbitrary μ with this at $\mu = 1$ and solving for the desired ratio:

$$\frac{C_\nu}{D_\nu} = \frac{\frac{I_\nu(\mu)}{I_\nu(1)} - \mu}{1 - \frac{I_\nu(\mu)}{I_\nu(1)}} \tag{24}$$

We can use this to calculate the ratio for each wavelength in Fig 14.16 of Allen's. The ratio will not be constant over μ for any given wavelength, so it is necessary to take the average values. The standard deviations are also calculated. The results are tabulated:

$\lambda\,(\mu\mathrm{m})$	C_ν/D_ν	StDev
0.2	0.65	0.20
0.5	0.43	0.14
1.0	1.7	0.45
2.0	3.2	1.1
5.0	7.7	3.2

We can do the same calculation for the grey atmosphere case. We do so by recalling that the temperature profile in the Eddington approximation is

$$T^4 = \frac{3}{4} T_{\mathrm{eff}}^4 \left(\tau + \frac{2}{3}\right) \tag{25}$$

so, with

$$S = aT^4 \tag{26}$$

we have

$$C = \frac{a}{2}T_{\rm eff}^4 \tag{27}$$

$$D = \frac{3a}{4}T_{\rm eff}^4 \tag{28}$$

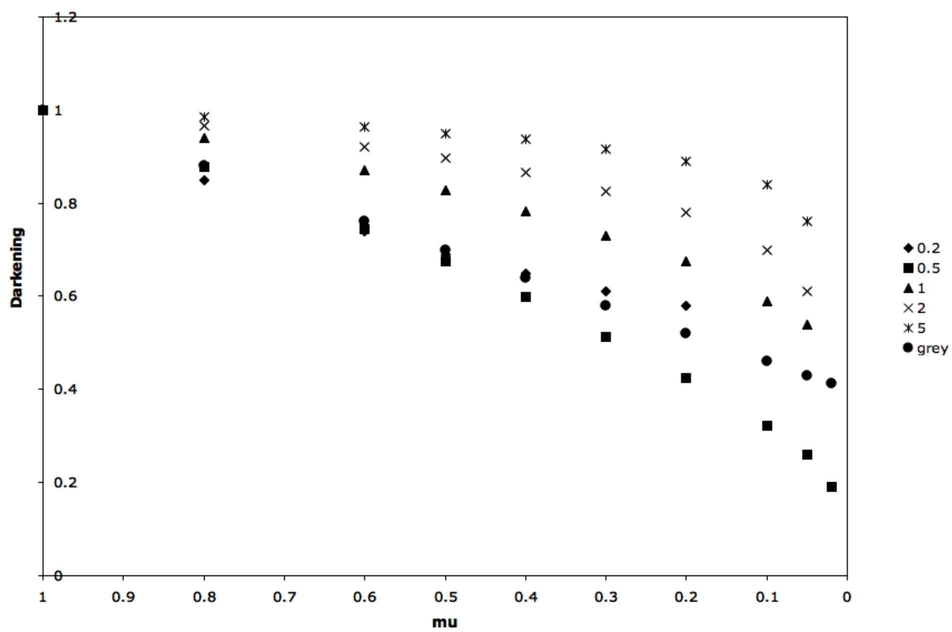
so

$$\frac{C}{D} = \frac{2}{3} \tag{29}$$

and

$$\frac{I(\mu)}{I(1)} = \frac{\mu + \frac{2}{3}}{1 + \frac{2}{3}} \tag{30}$$

This is plotted alongside the given limb darkening curves.



By comparison with the curves, 0.2 and 0.5 μm best represent the grey-body case. The other wavelengths do not resemble the grey case.

We see from the previous calculations that 0.2 and 0.5 μm have average C_ν/D_ν closest to the grey prediction, while 0.5 and 1 μm have the smallest fractional change. We thus conclude that $0.5\,\mu$m is the approximate wavelength of the Rosseland mean opacity.

2b:

At the centre of the disk, we see

$$I_\nu = C_\nu + D_\nu = S_\nu(\tau = 1) \tag{31}$$

Assuming, LTE, S_ν is due to blackbody radiation, so:

$$I_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{\exp(\frac{hc}{\lambda kT}) - 1} \tag{32}$$

where T is at $\tau = 1$. Solving for T:

$$T = \frac{hc}{\lambda k}\frac{1}{\ln\left(1 + \frac{2hc^2}{I_\lambda \lambda^5}\right)} \tag{33}$$

Calculating this for each of the wavelengths gives:

$\lambda\,(\mu\mathrm{m})$	I_λ (cgs)	$T(\tau_\lambda = 1)$ K
0.2	0.014×10^{10}	4900
0.5	3.63×10^{10}	6210
1.0	1.21×10^{10}	6060
2.0	0.18×10^{10}	6440
5.0	0.0057×10^{10}	5630

2c:

We now have $R = C_\nu / D_\nu$ and $C_\nu + D_\nu$ from

$$C_\nu + D_\nu = I_\nu = I_\lambda \frac{\lambda^2}{c} \tag{34}$$

so

$$C_\nu = \frac{I_\lambda \lambda^2 / c}{1 + R} \tag{35}$$

and

$$D_\nu = \frac{R I_\lambda \lambda^2 / c}{1 + R} \tag{36}$$

Plugging in the values, we get

$\lambda\,(\mu\mathrm{m})$	C_ν (cgs)	D_ν (cgs)
0.2	7.4×10^{-9}	1.1×10^{-8}
0.5	9.1×10^{-6}	2.1×10^{-5}
1.0	2.5×10^{-5}	1.5×10^{-5}
2.0	1.8×10^{-5}	5.6×10^{-6}
5.0	4.2×10^{-6}	5.5×10^{-7}

2d:

We know that

$$\tau_\nu = \frac{S_\nu(T) - C_\nu}{D_\nu} \tag{37}$$

Using $T = 6060\,\mathrm{K}$, corresponding to the physical depth of $\tau_\nu = 1$ for $1.0\,\mu\mathrm{m}$, we can then calculate the optical depth as a function of wavelength for this physical depth:

$\lambda\,(\mu\mathrm{m})$	τ_ν
0.2	28
0.5	0.85
1.0	1
2.0	0.66
5.0	1.8

2f:

Based on the above values, the opacity is greatest at $0.2\,\mu\mathrm{m}$. It decreases rapidly towards $0.5\,\mu\mathrm{m}$. It increases slightly near $1.0\,\mu\mathrm{m}$, but decreases even more at $2.0\,\mu\mathrm{m}$. It then begins to increase again towards $5.0\,\mu\mathrm{m}$.

Problem 3:

3a:

The Kurucz calculation for T v. τ is plotted against the grey case

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 (\tau + 2/3) \tag{38}$$

with $T_{\text{eff}} = 5770\,\text{K}$. Calculating the percent deviation from the Kurucz model for each point between $0.1 \leq \tau < 2$ gives a maximum deviation of 4% at $\tau = 0.75$ and $\tau = 1$.

See attached figure.

3b and 3c:

We calculate the limb darkening from

$$I(\mu) = \frac{1}{\mu} \int_0^\infty S \mathrm{e}^{-\tau/\mu} d\tau \tag{39}$$

and

$$S = aT^4 \tag{40}$$

We can integrate this numerically from Kurucz's model.

This is plotted alongside the curves for the grey case and measured curves from problem 2.

The Kurucz curve follows the grey curve and the $0.2\,\mu$m curve.

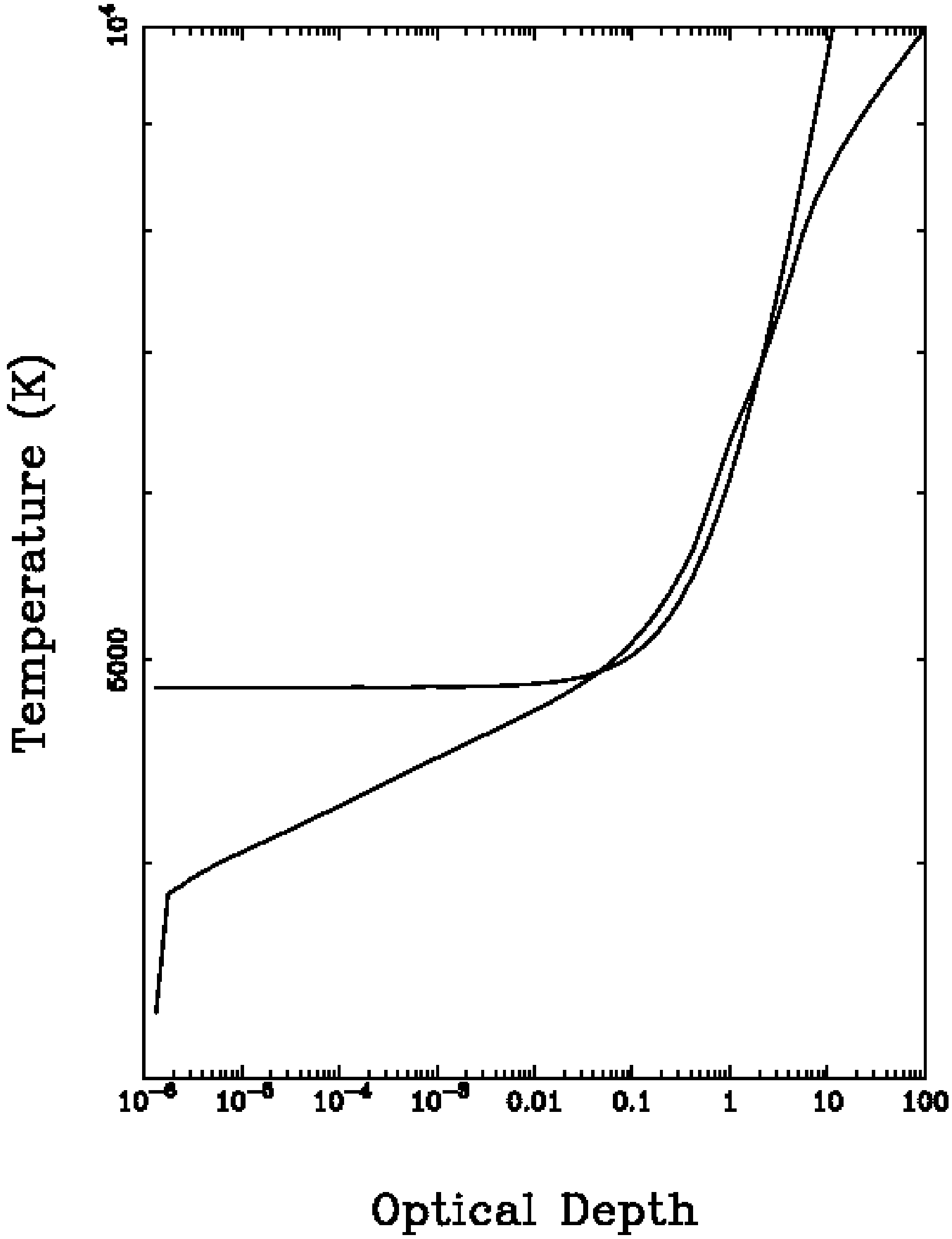
3d:

We are testing the approximation

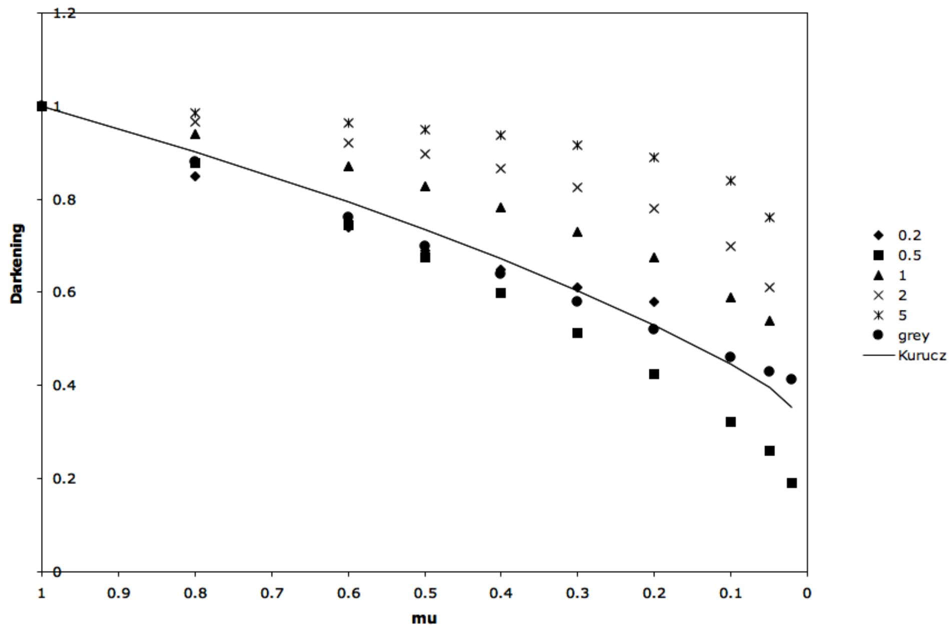
$$T(\tau) \simeq \frac{T_{\text{eff}}}{T_{\text{eff}}^0} T^0(\tau) \tag{41}$$

Using the simulation grid from file `ap00k4nover.dat` located in Kurucz's directory `GRIDP00NOVER`, we consider the stars will $T_{\text{eff}} = 4500 - 5500\,\text{K}$ and $\log(g) = 4.0$.

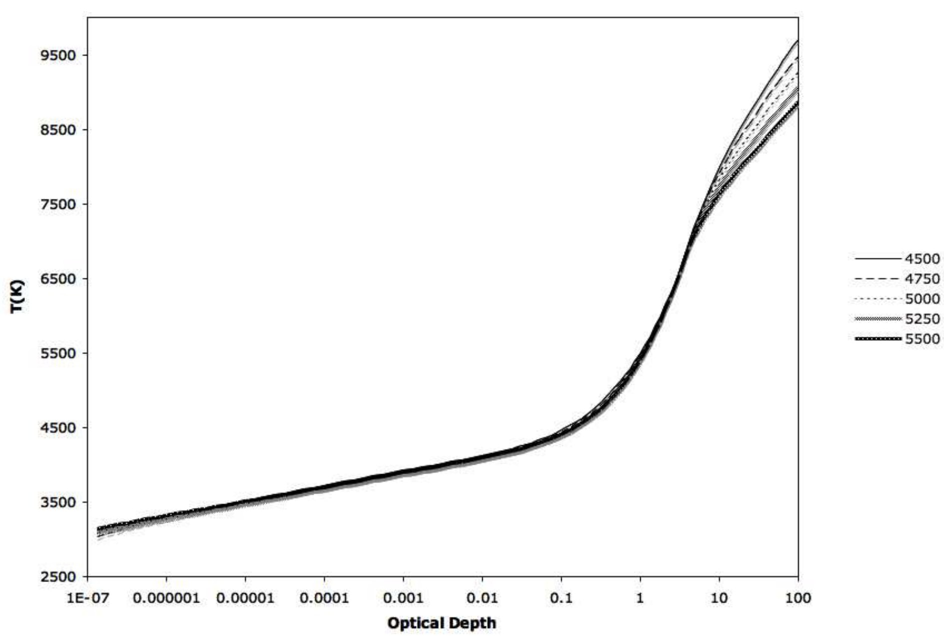
Converting the temperature profiles for each effective temperature to that of a star with $T_{\text{eff}} = 5000\,\text{K}$, a plot was created. As can be seen, the approximated profiles agree well for optical depths between $\tau = 10^{-6}$ and $\tau = 10$.



Temperature (K)
10⁴
5000
10⁻⁶
10⁻⁵
10⁻⁴
10⁻³
0.01
0.1
1
10
100
Optical Depth



Darkening
mu
0.2
0.5
1
2
5
grey
Kurucz



T(K)
Optical Depth
4500
4750
5000
5250
5500

Converting the 5000 K star to the other temperatures, the maximum deviations between optical depths of 0.1 and 2 are compiled. As can be seen, the agreement is quite nice.

$T_{\rm eff}$ K	Max Deviation (%)	τ
4500	1.2	0.56
4750	0.50	0.56
5250	0.40	0.56
5500	0.72	0.42