Ay 101 - Fall 2023

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Problem Set 5

due Friday, 10 November, 2023

This week we are putting it all together in terms of fundamental stellar properties, bringing in the various equilibrium conditions of the stellar structure equations, and considering fundamental stellar properties like mass and radius.

1. **Linear Density Model** Assume a star obeys a linear density model so that

$$\rho(r) = \rho_c \left(1 - \frac{r}{R_*}\right),$$

where ρ_c is the central density and R_* is the radius of the star. The nuclear production rate per unit mass is equal to

$$\epsilon(r) = \epsilon_c \left(1 - \frac{r}{0.2R_*}\right) \text{ for } r \leq 0.2R_* \text{ and}$$

$$\epsilon(r) = 0 \text{ for } r > 0.2R_*.$$

(a) Find an expression for the mass as a function of r/R_* in terms of the total mass of the star, M_*. It should have the form $M(x) = M_* f(x)$, where $x = r/R_*$ and $f(x)$ is some function you will determine.

(b) Using the equation of hydrostatic equilibrium, find the total pressure as a function of the fractional radius, r/R_*, in terms of R_* and M_*.

(c) Finally, find an expression for the total luminosity in terms of R_*, M_*, and ϵ_c.

(d) Assuming this is a valid model for our sun, use the established values of $R_\odot$, $M_\odot$, and $L_\odot$ to determine $\epsilon_{c,\odot}$.

2. **Polytropic Mass-Radius Relationship**

a. (Guidry problem 8.2) Apply pure scaling (dimensional) arguments to the equation for hydrostatic equilibrum to obtain directly the Lane-Emden result $M \propto R^{(3-n)/(1-n)}$ relating the mass M and radius R for a star with a polytropic equation of state $P = K\rho^{1+1/n} = K\rho^\gamma$, where K is a constant, P is the pressure, ρ is the mass density, and n is the polytropic index.

b. Guidry problem 8.8

c. For the typically considered cases of n=3 and n=3/2, what are the corresponding mass-radius relationships? Where along the main sequence would these relationships apply?

3. **Homology with the p-p Chain.** Consider a family of chemically homogeneous stars which are similar in every respect except for their masses and radii. The similarity of the stars is such that, for any member of the family with mass M and radius R, the density at distance r from the center can be written as a function of $x = r/R$ in the following way:

$$\rho(R) = \frac{M}{R^3} F_\rho(x),$$

where the function $F_\rho(x)$ is common to the entire family. In a similar way, the mass enclosed by a sphere of radius r within the star can be written as

$$m(r) = M F_m(x),$$

where, again, the function $F_m(x)$ is common to the family. Assume that the equation of state for the stellar material is the ideal classical gas equation, that the opacity of the material obeys Kramers' law ($\kappa = \kappa_0 \rho T^{-3.5}$), and that nuclear energy is generated by the proton-proton chain ($\epsilon = \epsilon_0 X^2 \rho T^4$).

   a. Use the fundamental equations of stellar structure to derive the following scaling relations for the pressure, the temperature, the power flow due to radiative diffusion and the power flow due to nuclear fusion:

$$P(r) = \frac{M^2}{R^4} F_\rho(x),$$

$$T(r) = \frac{M}{R} F_T(x),$$

$$L_{\rm rad}(r) = \frac{M^{5.5}}{R^{0.5}} F_{\rm rad}(x),$$

$$L_{\rm fus}(r) = \frac{M^6}{R^7} F_{\rm fus}(x),$$

   where, again, the functions $F(x)$ are common to the family.

   b. Show that this family of stars will lie along a line on the Hertzsprung-Russell diagram of L, luminosity, vs T_{eff}, effective surface temperature, given by

$$L \propto T_{eff}^{4.12}.$$

4. **Energy Generation.** Assume that both the main *pp* chain and the main *CNO* cycle reactions are operating, and that they are in equilibrium. Use an estimate of the central temperature of a star as a function of its mass to:

a. Predict the total stellar energy generation rate in the center of the star via nuclear reactions as a function of stellar mass from 0.1 to 100 Solar masses.

b. Predict the fraction of energy coming from the *pp* chain versus that from the *CNO* cycle in the center of the star as a function of stellar mass. This is perhaps appreciated best by looking at $\epsilon_{pp}/\epsilon_{total}$ and $\epsilon_{CNO}/\epsilon_{total}$ as well as the ratio $\epsilon_{pp}/\epsilon_{CNO}$.

c. Plot your results for part a and part b as a function of stellar mass.

5. **Term Project.** Continue the Ay101 Term Project, moving on to the "Weeks 7-8-9" portion. This is due at the end of the term, but please try to make progress each week.

[for all assignments, please write near your name how many hours you spent on the set.]