States of Matter

(subtitle: more stat mech. and more rho vs T)
It is standard practice to introduce three \textit{adiabatic exponents} $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ through

\[
\Gamma_1 \equiv \left( \frac{\partial \ln P}{\partial \ln \rho} \right)_S, \quad \frac{\Gamma_2}{\Gamma_2 - 1} \equiv \left( \frac{\partial \ln P}{\partial \ln T} \right)_S, \quad \Gamma_3 - 1 \equiv \left( \frac{\partial \ln T}{\partial \ln \rho} \right)_S
\]

where the subscripts $S$ remind us that adiabatic processes correspond to constant entropy, and where the logarithmic derivatives are equivalent to $\partial \ln A = \partial A / A$. This implies equations of state having one of the three forms

\[
P V^{\Gamma_1} = c_1 \quad P^{1-\Gamma_2} T^{\Gamma_2} = c_2 \quad T V^{\Gamma_3-1} = c_3.
\]

where the $c_n$ are constants.

From the above definitions,

\[
\Gamma_1(\Gamma_2 - 1) = \Gamma_2(\Gamma_3 - 1),
\]

so the $\Gamma$ are not independent.
For the special case of ideal gases

\[ \Gamma_1 \equiv \left( \frac{\partial \ln P}{\partial \ln \rho} \right)_S \]
\[ \frac{\Gamma_2}{\Gamma_2 - 1} \equiv \left( \frac{\partial \ln P}{\partial \ln T} \right)_S \]
\[ \Gamma_3 - 1 \equiv \left( \frac{\partial \ln T}{\partial \ln \rho} \right)_S \]

are equal and equivalent to the ideal gas \( \gamma \),

\[ \Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma \quad \text{(ideal gas)}. \]

But for more general equations of state \( \Gamma_1, \Gamma_2, \) and \( \Gamma_3 \) are distinct and carry information emphasizing different aspects of the gas thermodynamics:

1. Because it relates \( \Delta P \) to \( \Delta \rho \), \( \Gamma_1 \) enters into dynamical properties of the gas like sound speed.

2. \( \Gamma_2 \) is important for convective gas motion, because it relates \( \Delta P \) to \( \Delta T \).

3. \( \Gamma_3 \) influences the response of the gas to compression, since it depends on the relationship of \( \Delta T \) to \( \Delta \rho \).
Fig. 2-13  The adiabatic exponents of a pure hydrogen gas as a function of its degree of ionization. Only the initial 50 percent is shown because the second 50 percent is its mirror image. The exponents change rapidly in the regions 0 to 1 or 99 to 100 percent ionization. Between 5 and 95 percent ionization, the values are considerably less than $\frac{3}{2}$ and therefore have a destabilizing influence on the structure. Numerical values are given in Table 2-4.
Maxwell-Boltzmann Law

When particles interact sufficiently they spread the available energy around and an equilibrium distribution evolves which means that the velocities of particles in thermodynamic equilibrium in a hot gas are distributed with a Gaussian or Maxwellian velocity distribution.

\[
p(v)dv = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT}} 4\pi v^2 dv
\]
Figure 4.5  The Maxwell distribution for H and C atoms in a gas at 10000 K. The units of \( f \) are the reciprocal of those of speed because \( f(V)\,dV \) has no units since it is a probability.
Fig. 15.1 For an electron gas with \( n_e = 10^{28} \text{ cm}^{-3} \) (corresponding to a density of \( \rho = 1.66 \times 10^4 \text{ g cm}^{-3} \) for \( \mu_e = 1 \)), the Boltzmann distribution function \( f(p) \) is shown by thin lines over the absolute value of the momentum \( p \) (both in cgs units) for three different temperatures (in K). The heavy line shows the parabola that gives an upper bound to the distribution function owing to the Pauli principle. (Note that the coordinates are not logarithmic but linear as in Figs. 15.2 and 15.5)

Fig. 15.5. The solid line gives the distribution function \( f(p) \) and \( p \) in cgs for a partially degenerate electron gas with \( n_e = 10^{28} \text{ cm}^{-3} \) and \( T = 1.9 \times 10^7 \text{ K} \), which corresponds to a degeneracy parameter \( \psi = 10 \) (cf. the case of complete degeneracy of Fig. 15.2). The dot-dashed line shows the further increase of the parabola that defines an upper bound for the distribution function.
Figure 3.4: Fermi surface in a degenerate fermi gas. The solid line defines a step function corresponding to the limit $T \to 0$ of the Fermi–Dirac distribution. Successively longer dashes represent the distribution at successively higher temperatures.

As illustrated in Fig. 3.4, in the limit that the temperature may be neglected the Fermi–Dirac distribution becomes a step function in energy space,

$$f_i(e_p) = \frac{1}{e^{(e_p - \mu)/kT} + 1} \quad \xrightarrow{T\to 0} \quad \begin{cases} f(e_p) = 1 & e_p \leq e_f \\ f(e_p) = 0 & e_p > e_f \end{cases}$$

- The value of the chemical potential $\mu$ at zero temperature is denoted by $e_f$ and is termed the fermi energy.
- The corresponding value of the momentum is denoted by $p_f$ and is termed the fermi momentum.
- Thus, the fermi energy gives the energy of the highest occupied state in the degenerate fermi gas.
Maxwellian form (ignore the 1) appropriate for ideal gas.

Fermi-Dirac form (the plus 1) appropriate for electrons

Bose-Einstein form (the minus 1) appropriate for photons

Fig. 1-3 Schematic occupation index. The Maxwell-Boltzmann distribution, which is a pure exponential curve at all temperatures and all values of $\epsilon$, is valid only if $\alpha \gg 0$. The Fermi-Dirac distribution has, because of the Pauli exclusion principle, an upper bound of unity on the occupation index. This upper bound is approached at low energy and low temperature (or high density). The Einstein-Bose distribution for photons has a large occupation index for $\epsilon \ll kT$ and an exponentially decreasing index for $\epsilon \gg kT$
Figure 3.3: Illustration of classical and quantum gases. In the classical gas the average spacing $d$ between gas particles is much larger than their deBroglie wavelengths $\lambda$. In the quantum gas the interparticle spacing is comparable to or less than the deBroglie wavelength. The gas particles have a range of deBroglie wavelengths because they have a thermal velocity distribution.
3.7.5 Summary: High Gas Density and Stellar Structure

This preceding discussion implies that increasing the density can have a large impact on the structure of stars. Generally, we may identify several important consequences of high densities in stellar environments:

- An increase in the gas density above a critical amount enhances the probability for pressure ionization, thereby creating a gas of electrons and ions irrespective of possible thermal ionization.

- An increase in the gas density, by uncertainty principle arguments, increases the average momentum of gas particles and makes them more relativistic.

- An increased density raises the fermi momentum. This, for example, influences the weak interaction processes that can take place in the star.

- An increase in the gas density decreases the interparticle spacing relative to the average deBroglie wavelength, making it more likely that the least massive particles in the system make a transition from classical to degenerate quantum gas behavior.
• Increased density *enhances the strength of the gravitational field* and makes it more difficult to maintain stability of the star against gravitational collapse. Higher density also makes it more likely that general relativistic corrections to Newtonian gravitation become important.

• Higher density (often implying higher temperature) tends to *change the rates of thermonuclear reactions* and to *alter the opacity of the stellar material* to radiation.
  
  – The former changes the rate of energy production;
  – the latter changes the efficiency of how that energy is transported in the star.

Both can have large consequences for stellar structure and evolution.
Oscillation, Instability, Pulsation
Categories of Large-Amplitude Variables

1. *Eclipsing binaries* are binary stars in which the total light output of the system is altered by geometrical eclipses of one star by the other. If the binary system is too far away to resolve the two components, this will appear to be a single star with periodic variation in light output.

2. *Eruptive and exploding variables* are stars that suddenly increase light output and often eject mass because of a rapid and violent disruption or partial disruption of the star. Novae and supernovae are dramatic examples in this category.

3. *Pulsating variable stars* appear to undergo (possibly complex) pulsations that alter the light output in periodic or irregular fashion, without disrupting significantly the overall structure of the star. Well-known examples of this category are Cepheid variables and RR-Lyrae stars.
The Zoo of Stellar Oscillations and Pulsations
Observed Power Spectrum of the Sun

“5 minute oscillations” (actually 3-8 min) discovered at CIT in 1962 – Leighton, Noyes

Figure 2.14: Power spectrum of solar oscillations, obtained from Doppler observations in light integrated over the disk of the Sun. The ordinate is normalized to show velocity power per frequency bin. The data were obtained from six observing stations and span approximately four months. Panel (b) provides an expanded view of the central part of the frequency range. Here some modes have been labelled by their degree $l$, and the large and small frequency separations $\Delta \nu$ and $\delta \nu_1$ [cf. equations (2.40) and (2.41)] have been indicated. (See Elsworth et al. 1995.)

Amplitudes: 15 cm/s in radial velocity
4 ppm in brightness
Simple Radial Pulsation Modes

- The *fundamental mode* has no nodes (points of zero motion) between the center and surface, implying that the stellar matter involved in the vibration all moves in the same direction at a given time.

- The *first overtone* has one node between the center and the surface, so the matter moves in one direction outside this node and in the opposite direction inside this node at a given phase of the pulsation.

- Likewise, higher overtones with additional nodes and more complex motion may be defined.

- Just as for musical instruments and other acoustically vibrating systems, a star may exhibit several modes of oscillation at once.

- The physical motion of the gas in radial stellar pulsations is largest in the fundamental mode and is considerably smaller in the first overtone.

- In higher overtones the motion of the gas in an oscillation cycle is even smaller.
Radial $p$-modes
(from outer zones)

Non-Radial $g$-modes
(from inner zones)
Oscillation Nomenclature

- $p_{nml}$
- $n$: number of radial nodes
- $l$: number of nodal lines
- $m$: $-l < m < l$
- $|m|$: number of nodes passing through poles
- $l-|m|$: number of nodes parallel to equator
- Related to spherical harmonics $Y_l^m(\theta, \phi)$
Spherical Harmonics $Y_{l,m}(\theta, \varphi)$

- ignoring phase factor $(-1)^m$
- $\theta=0$ axis is inclined 45º towards you
- each panel labeled $l,m$
- $l$ defines horizontal wavenumber $kh = L/R$, where $L = \left[ l/(l+1) \right]^{0.5}$ and surface wavelength $2\pi/kh$
- $m$ gives phase $-1 < m < l$ ($2l+1$ values)

Christensen-Dalsgaard
Non-radial Oscillations

- $l=6, m=0$
- $l=6, m=3$
- Solar $p$ modes
  - $l=20, m=16, n=14$
Modes characterized by frequency $\nu$ and angular degree $l$

- $l=0$ are radial modes; can detect $l<3$ in integrated light
- high frequency waves are acoustic or $p$-modes
- low frequency waves are buoyancy or $g$-modes; valuable probes of stellar interior
- *intermediate frequency* $f$-modes are surface versions of $g$-modes

In the Sun we detect $l = 0$ to 1000 and $g = 1$ to 4 !

In other stars, usually $l < 3$ and no $g$.

Christensen-Dalsgaard
Model Predictions of $\nu, l$ distribution

Solar Data

Christensen-Dalsgaard
Refraction and Reflection of Acoustic Waves

- p-mode waves can be refracted and reflected by the change in $c_S$ and density with depth
- effect more pronounced for large angular degrees $l$ (marked in red)
- surface reflection due to abrupt density change
- internal effects due to changes in $c_S$
- high frequency modes probing deep interior can discern the rapid density change at edges of the convective zones ➞ their sizes!
Figure 2. The pattern of oscillation frequencies seen in the Sun observed as a star, showing the characteristic large and small frequency spacings that indicate the mean density and age (adapted from Elsworth & Thompson 2004).
Inferred Sound Speed
(Deviations from Model Sun)

\[ \frac{\delta c^2}{c^2} \]

Christensen-Dalsgaard
Inferred Internal Rotation vs Latitude!

The pulsational modes are split by rotation.

![Graph](image-url)

- Rotation Rate, nHz
- $r/R$
- Convection zone
- $0^\circ$, $30^\circ$, $60^\circ$
Figure 1. Power spectra of time series observed in radial velocity for the four stars \( \xi \) Hydrae (G7 III; Frandsen et al. 2002), Procyon (F5 IV; Arentoft et al. 2008), \( \alpha \) Centauri A (G2 V; Bedding et al. 2004) and \( \alpha \) Centauri B (K2 V; Kjeldsen et al. 2005). The oscillation periods for the four stars are: around 3–4 hours for \( \xi \) Hydrae, 20–25 minutes for Procyon, 7 minutes for \( \alpha \) Cen A and 4 minutes for \( \alpha \) Cen B. These differences, and those in the detailed structure of the power spectra, reflect the differences in stellar properties (radius, mass, surface temperature and age). Note that the vertical scales are normalized – the actual amplitudes decrease from top to bottom in the figure.
Figure 2.24: Power spectrum of the DB variable GD358, obtained with the
Figure 7.2: Evolution tracks (---) and curves of constant central hydrogen abundance (-----) in ($\Delta \nu_0$, $D_0$) diagrams. Here $\Delta \nu_0$ is the average separation between modes of the same degree and adjacent radial order, and $D_0$ is related to the small separation between $\nu_{nl}$ and $\nu_{n-1l+2}$ (cf. eq. 7.58). The stellar masses, in solar units, and the values of the central hydrogen abundance, are indicated. In panel (b), the frequency separations have been scaled by $(\bar{\rho})^{-1/2}$ ($\bar{\rho} \propto M/R^3$ being the mean density), to take out the
Stars that pulsate at large amplitude are concentrated in a particular zone of the HR diagram.

Cooling from left to right (post-MS for Cepheids, and post-He burning for RR Lyrae), the ionization zones (where $\gamma$ drops) shift from the outer envelope with little mass to the inner core.

The transition zone where pulsations are allowed by the $\kappa$-mechanism is the so-called ‘instability strip’

The period-luminosity relation follows from $\tau \sim \rho^{-1/2}$
FIGURE 14.14  Hydrogen and helium ionization zones in stars of different temperatures. For each point in the star, the vertical axis displays the logarithm of the fraction of the star’s mass that lies above that point.
Temperature Boundaries for the Instability Strip

The radial location of hydrogen and helium ionization zones in stars of particular surface temperatures, and onset of convection near the surface for stars with surface temperatures that are too low, are determining factors in producing the instability strip.

- The physical radius for the hydrogen and helium partial ionization zones within a given star will depend strongly on the effective surface temperature of that star.

- For stars with higher temperatures, ionization zones will be near the surface and there will be insufficient mass in the partially-ionized layers to drive sustained oscillations.

- If the surface temperature is too low, convection in the outer layers will undermine the \( \kappa \)-mechanism (detailed simulations show that convection interferes with the trapping effect and thus damps stellar pulsations).

- This suggests an optimal range of surface temperatures for which

  1. the ionization zones are deep enough to drive sustained oscillations by coupling to the fundamental and overtones of the characteristic vibrational frequencies (\( \rightarrow \) higher-temperature end of the optimal range),

  2. but for which the convection is not strong enough to invalidate the mechanism (\( \rightarrow \) lower-temperature end of the optimal range).
12.3. NON-ADIABATIC RADIAL PULSATIONS

![Graph showing variations in magnitude, temperature, spectral class, radial velocity, and phase.](image)

Guidry
The Cepheid period-luminosity relation has been central in establishing the cosmic distance scale since:

- Cepheids are luminous variables
- Their τ-L relation can be calibrated via examples in the Milky Way
- They can be found in other galaxies and hence used to establish the rate of the Hubble expansion.