AY 123 Solutions Pset 1

1.a. $dM(r) = 4\pi r^2 \rho(r)\, dr$

$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right)$

$dM(r) = 4\pi r^2 \rho_c \left(1 - \frac{r}{R}\right) dr$

$M(r) = \int_0^r 4\pi r^2 \rho_c \left(1 - \frac{r}{R}\right) dr$

$M(r) = \frac{4\pi}{3} \rho_c r^3 \left(1 - \frac{3r}{4R}\right)$

@ center, $r = R$

$\Rightarrow M(R) = \frac{4\pi}{3} \rho_c \left(1 - \frac{3}{4}\right) = \frac{\pi}{3} \rho_c R_*^3$

$$\boxed{\rho_c = \frac{3M}{\pi R^3}}$$

b. $\frac{dP(r)}{dr} = -\frac{\rho(r)\, G M(r)}{r^2}$

$M(r) = \frac{4\pi}{3} \rho_c r^3 \left(1 - \frac{3r}{4R}\right)$

$\Rightarrow \frac{dP(r)}{dr} = -\rho_c \left(1 - \frac{r}{R}\right) G \cdot \frac{4\pi}{3} \rho_c r^3 \left(1 - \frac{3r}{4R}\right) =$

$= \frac{4\pi}{3} G \rho_c^2 r \left(1 - \frac{r}{R}\right)\left(1 - \frac{3r}{4R}\right)$

1.b. (cont.) $\int_{P_c}^{P(r)} dP = \int_0^r -\frac{4}{3}\pi G \rho_c^2 r \left(1-\frac{r}{R}\right)\left(1-\frac{3r}{4R}\right) dr$

$$P(r) = P_c - \frac{4\pi}{3} G \rho_c^2 \left(\tfrac{1}{2} r^2\right)\left(1 - \frac{7r}{6R} + \frac{3r^2}{8R^2}\right)$$

at $r=R$, $P(R)=0$

$$\Rightarrow 0 = P_c - \frac{4\pi G}{6}\left(\frac{3M}{\pi R^3}\right)^2 R^2 \left(1 - \frac{7}{6} + \frac{3}{8}\right)$$

$$P_c = \frac{2\pi G}{3} \cdot \frac{9M^2}{\pi^2 R^6} R^2 \left(\frac{5}{24}\right) = \boxed{\frac{5GM^2}{4\pi R^4} = P_c}$$

$$P(r) = \frac{5GM^2}{4\pi R^4}\left(1 - \frac{24}{5R^2} r^2 \left(1 - \frac{7r}{6R} + \frac{3r^2}{8R^2}\right)\right)$$

$$\boxed{P(r) = P_c \underbrace{\left(1 - \frac{24}{5}\frac{r^2}{R^2} + \frac{28}{5}\frac{r^3}{R^3} - \frac{9}{5}\frac{r^4}{R^4}\right)}_{f\left(\frac{r}{R}\right)}}$$

$$\boxed{P_c = \frac{5GM_\odot^2}{4\pi R_\odot^4} = 4.38 \times 10^{15} \left[\frac{M}{M_\odot}\right]^2 \left[\frac{R}{R_\odot}\right]^{-4}}$$

c. $P = \frac{\rho k T}{\mu m_H} \longrightarrow T_c = \frac{P_c \mu m_H}{\rho_c k}$

1.c. (cont.) $T_c = \left(\frac{5GM^2}{4\pi R^4}\right) \frac{\mu m_H}{K} \left(\frac{\pi R^3}{3M}\right) = \frac{5GM\mu m_H}{12KR}$

$$\boxed{T_c = \frac{5GM\mu m_H}{12KR}}$$

d. $P_{rad} = \frac{4\sigma T_c^4}{3c}$ $\quad P_{gas} = \frac{\rho_c K T_c}{\mu m_H}$

$$\frac{P_{rad}}{P_{gas}} = \frac{4\sigma}{3c}\left(\frac{5GMm_H\mu}{12KR}\right)^4 \left(\frac{\mu m_H}{K}\left(\frac{\pi R^3}{3M}\right)\left(\frac{12KR}{5GMm_H\mu}\right)\right)$$

$$\frac{P_{rad}}{P_{gas}} = \frac{4\sigma}{9c}\left(\frac{5}{12}\right)^3\left(\frac{m_H\mu}{K}\right)^4 G^3 M^2 = 3.1\times10^{-4}\left[\frac{M}{M_\odot}\right]^2$$

When P_{gas} is comparable to P_{rad}, $\frac{P_{rad}}{P_{gas}} = 1$

$$\Rightarrow \frac{P_{rad}}{P_{gas}} = 3.1\times10^{-4}\frac{M^2}{M_\odot^2} = 1$$

$$\boxed{M = 56.8\,M_\odot}$$

e. $\Omega = -G\int \frac{M(r)}{r}\,dM(r)$

$$= -G\int \frac{\left(\frac{4\pi}{3}\rho_c r^3\right)\left(1-\frac{3r}{4R}\right)}{r} 4\pi r^2 \rho_c \left(1-\frac{r}{R}\right)dr$$

1.e. (cont.) $\boxed{\Omega = -\frac{26}{35}\frac{GM^2}{R}}$

Now show that $2U = -\Omega$

$$U = -\frac{3}{2}\int P\,dV = -\frac{3}{2}\int \frac{P(r)\,dM(r)}{\rho(r)}$$

$$dM(r) = 4\pi r^2 \rho(r)\,dr$$

$$\Rightarrow U = -\frac{3}{2}\int P(r)\frac{4\pi r^2 \cancel{\rho(r)}}{\cancel{\rho(r)}}dr = -\frac{3}{2}\int 4\pi r^2 P(r)\,dr$$

From part b) plug in for $P(r)$

$$U = -6\pi\int r^2 \frac{5GM^2}{4\pi R^4}\left[1 - \frac{24}{5}\frac{r^2}{R^2} + \frac{28}{5}\frac{r^3}{R^3} - \frac{9}{5}\frac{r^4}{R^4}\right]dr$$

$$U = \frac{13}{35}\frac{GM^2}{R}$$

$$\left(\frac{13}{35}\frac{GM^2}{R}\right)2 = \frac{26}{35}\frac{GM^2}{R} = -\Omega$$

$$\Rightarrow \boxed{2U = -\Omega}$$

Problem 2

The sound speed is given by:

$$v_s^2 = \gamma \frac{P}{\rho}$$

Using the Virial theorem:

$$\Omega = -3 \int P\, dV \approx -\frac{GM^2}{R}$$

$$dV = \frac{dm}{\rho}$$

so $$3 \int \frac{P}{\rho} dm = \frac{GM^2}{R}$$

$$3 \int \frac{v_s^2}{\gamma} dm = \frac{GM^2}{R}$$

assuming v_s and γ are $\sim$ constant, then

$$3 \frac{v_s^2}{\gamma} M = \frac{GM^2}{R}$$

$$\boxed{v_s = \left(\frac{\gamma G M_\odot}{3 R_\odot}\right)^{1/2} \approx 3 \times 10^7 \text{ cm/s for } \gamma = 5/3}$$

$$P = \frac{2R}{v_s} = \frac{2R_\odot}{\sqrt{\frac{\gamma G M_\odot}{3 R_\odot}}} = \frac{2\sqrt{3} R_\odot^{3/2}}{\sqrt{\gamma G M_\odot}}$$

$$\rho_\odot = \frac{3 M_\odot}{4\pi R_\odot^3}$$

$$\frac{R_\odot^{3/2}}{M^{1/2}} = \sqrt{\frac{3}{4\pi \rho_\odot}}$$

so $$\boxed{P = \sqrt{\frac{9}{4\pi\gamma}} \frac{2}{\sqrt{G\rho_\odot}} \approx (G\rho_\odot)^{-1/2} \approx 1 \text{ hr}}$$

$$P \propto \rho^{-1/2} \propto \frac{R^{3/2}}{M^{1/2}}$$

$L \propto R^2 T_{eff}^4 \propto R^2$ for fixed T_{eff}

so $R \propto L^{1/2}$ and $R^{3/2} \propto L^{3/4}$

$L \propto M^3$, so $M \propto L^{1/3}$ and $M^{1/2} \propto L^{1/6}$

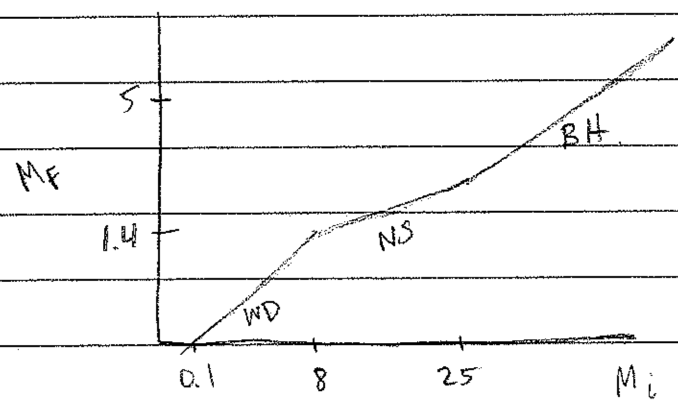
Putting this all together, $\boxed{P \propto \frac{L^{3/4}}{L^{1/6}} \propto L^{7/12}}$

3. Initial masses

White dwarfs : $0.1 - 8 M_\odot$
Neutron stars : $8 \simeq 25 M_\odot$
Black holes : $\sim 25 - 80 M_\odot$

Final masses

White dwarfs : $0.1 \sim 1.4 M_\odot$
Neutron Stars : $\sim 0.8 \sim 2.5 M_\odot$
Black holes : $\sim 6 - 10 M_\odot$



Assume stars formed between the initial masses of $0.1 M_\odot$ and $80 M_\odot$
Let the IMF have normalization constant a. Then

$$\xi(M) = a M^{-3.5}$$

The population will then have a total initial mass of

$$M_{tot} = \int_{0.1 M_\odot}^{80 M_\odot} M_i \, \xi(M_i) \, dM_i$$

3. (cont.) $M_{tot} = \int_{0.1}^{80} a M_i^{-1.35} dM_i = 5.71a$

In 10 Gyrs, which stars will have evolved off the MS (and contributed to the mass locked up in remnants)?

$$t_{MS} \cong 10^{10} \left(\frac{M_*}{M_\odot}\right)\left(\frac{L_*}{L_\odot}\right)^{-1}$$

$$L \propto M^{3.5}$$

$$t_{MS} \sim 10^{10} \left(\frac{M_*}{M_\odot}\right)\left(\frac{M_\odot^{3.5}}{M_*^{3.5}}\right) = 10^{10}\left(\frac{M_\odot^{2.5}}{M_*^{2.5}}\right)$$

$\Rightarrow$ In 10 Gyrs, all stars with $M \gtrsim 1 M_\odot$ will have evolved off the main sequence.

What is the mass locked up in low mass MS stars and stellar remnants?

$$M_{tot,remain} = \int_{0.1M_\odot}^{1M_\odot} M_i \xi(M_i) dM_i + \overbrace{\int_{1M_\odot}^{8M_\odot} \frac{M_i}{5} \xi(M_i) dM_i}^{WD} + \int_{8M_\odot}^{25M_\odot} \frac{M_i}{10} \xi(M_i) dM_i$$

~slope of line

$$= \int_{0.1}^{1M_\odot} a M_i^{-1.35} dM_i + \int_{1M_\odot}^{8M_\odot} \frac{a M_i^{-1.35}}{5} dM_i + \int_{8M_\odot}^{25M_\odot} \frac{a M_i^{-1.35}}{10} dM_i$$

$$= 3.54a + 0.30a + 0.05a = 3.89a$$

3. (cont.). The amount returned to the ISM is:

$$5.71a - 3.89a = 1.82a$$

The fraction of the mass of a stellar population that has been returned to the ISM after 10 Gyrs is therefore:

$$\frac{1.82a}{5.71a} = \boxed{32\%}$$

Problem 4

$\gamma = \frac{n+1}{n}$, so $\gamma = 1 \Rightarrow \boxed{n = \infty}$

Following the derivation of the Lane-Emden Equation from class,

start with hydrostatic equilibrium

$$\frac{dP}{dr} = \frac{-GM\rho}{r^2}$$

$$M = \frac{-r^2}{G\rho}\frac{dP}{dr}$$

$$\frac{dM}{dr} = \frac{-1}{G}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right)$$

setting this equal to the mass conservation equation,

$$\frac{-1}{G}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = 4\pi r^2 \rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \quad (*)$$

using the polytropic equation of state,

$P = K\rho$ where K is a constant

so $\frac{dP}{dr} = \frac{K d\rho}{dr}$

plugging this into $(*)$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho} K\frac{d\rho}{dr}\right) = -4\pi G\rho$$

now use $\rho = \rho_c e^{-\psi}$, $\frac{d\rho}{dr} = -\rho_c e^{-\psi}\frac{d\psi}{dr} = -\rho\frac{d\psi}{dr}$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho} K\left(-\rho\frac{d\psi}{dr}\right)\right) = -4\pi G\rho_c e^{-\psi}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 K\frac{d\psi}{dr}\right) = 4\pi G\rho_c e^{-\psi}$$

$$\underbrace{\frac{K}{4\pi G\rho_c}}_{\alpha^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = e^{-\psi}$$

set $r = \alpha\xi$, $\frac{d}{dr} = \frac{1}{\alpha}\frac{d}{d\xi}$

so $\alpha^2 \frac{1}{\alpha^2 \xi^2} \frac{1}{\alpha} \frac{d}{d\xi}\left(\alpha^2 \xi^2 \frac{d\psi}{d\xi} \frac{1}{\alpha}\right) = e^{-\psi}$

simplifying gives $\boxed{\frac{1}{\xi^2} \frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right) = e^{-\psi}}$

The surface of the star is defined to be the location where P and $\rho \to 0$

$\rho = \rho_c e^{-\psi(\xi)}$ will not go to 0 for any finite value of ψ or ξ, assuming ψ is a well-behaved function, so the star has no finite surface, and is therefore unphysical

An equation of state $P \propto \rho \propto e^{-\psi}$ can be used for astrophysical objects with no well-defined boundary. Examples include star and galaxy clusters, dark matter halos, and gas in star forming regions and galaxy clusters. These systems can be considered isothermal, and have no single boundary or edge.