1. Look at the total thermal energy of the system, $E_{tot} = \Omega + U$

$$\Omega \propto \frac{1}{R} \Rightarrow \Omega = -aR^{-1}$$

$$T = \frac{P \mu m_H}{\rho k} \qquad P \propto \rho^{\gamma}$$

$$\Rightarrow T \propto \frac{\rho^{\gamma-1} \mu m_H}{k} \qquad \rho \propto \frac{M}{R^3}$$

$$T \propto \left(\frac{M}{R^3}\right)^{\gamma-1} = M^{\gamma-1} R^{3(1-\gamma)}$$

$$\Rightarrow T \propto R^{3(1-\gamma)} \propto U$$

$$U = b R^{3(1-\gamma)}$$

$$E_{tot} = bR^{3(1-\gamma)} - aR^{-1}, \text{ where } a, b > 0$$

Consider three cases:

$\gamma > \frac{4}{3}$: $E = bR^{\alpha} - aR^{-1}$, where $\alpha < -1$. The potential energy term dominates at large radii, while the kinetic term dominates at small radii. Thus you get stability when the energy rises to both large and small radii.

$\gamma = \frac{4}{3}$: $E = (b-a)R^{-1}$. The smaller R is, the lower E, so

1. (cont.) the most stable configuration is a collapse to zero size (or more realistically, until other physics takes over).

$\gamma < \frac{4}{3}$: $E = bR^{\alpha} - aR^{-1}$, where $\alpha > -1$. The potential energy term dominates at small radii, while the kinetic term dominates at large radii. At large radii the system is unbound; at small radii, you get an indefinite collapse.

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Could also start from the Virial Thm:

$$E_{kin} = \frac{3}{2} \int \frac{P}{\rho} dM$$

$$\propto \int \rho^{\gamma-1} dM \quad \propto \bar{\rho}^{\gamma-1} \quad \text{at fixed } M, \bar{r}(m)$$

$$\bar{\rho} \propto R^{-3} \quad \text{at fixed } M$$

$$E_{kin} \propto R^{3(1-\gamma)} \quad \text{at fixed } M, \bar{r}(M)$$

2. Boundary for instability: $\left.\frac{dT}{dr}\right|_{ad} = -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM}{r^2}$

$$\int_T^{T_s} dT = \int_R^{R_s} -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM}{r^2}\,dr$$

$$T\Big|_T^{T_s} = +\left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{K}\frac{GM}{r}\Big|_R^{R_s}$$

$$T_s - T = \left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{K}GM\left(\frac{1}{R_s}-\frac{1}{R}\right)$$

For $\gamma = \frac{4}{3}$

$$\Rightarrow \boxed{T = \frac{GM\mu m_H}{4K}\left(\frac{1}{R}-\frac{1}{R_s}\right)+T_s}$$

3. (i). $B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$

$$\frac{\partial B_\nu}{\partial T} = \frac{2h\nu^3}{c^2} \frac{h\nu}{kT^2} \exp\left(\frac{h\nu}{kT}\right) \frac{1}{\left(\exp\left(\frac{h\nu}{kT}\right) - 1\right)^2}$$

$$\boxed{\frac{\partial B_\nu}{\partial T} = \frac{2h^2\nu^4}{c^2 kT^2} \frac{\exp\left(\frac{h\nu}{kT}\right)}{\left(\exp\left(\frac{h\nu}{kT}\right) - 1\right)^2}}$$

(ii). $x = \frac{h\nu}{kT}$

$$\frac{\partial B_\nu}{\partial T} = \frac{2k^3T^2}{h^2c^2} \frac{h^4\nu^4}{k^4T^4} \frac{\exp\left(\frac{h\nu}{kT}\right)}{\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^2}$$

$$\boxed{\frac{\partial B_\nu}{\partial T} = \frac{2k^3T^2}{h^2c^2} \frac{x^4 e^x}{(e^x - 1)^2}}$$

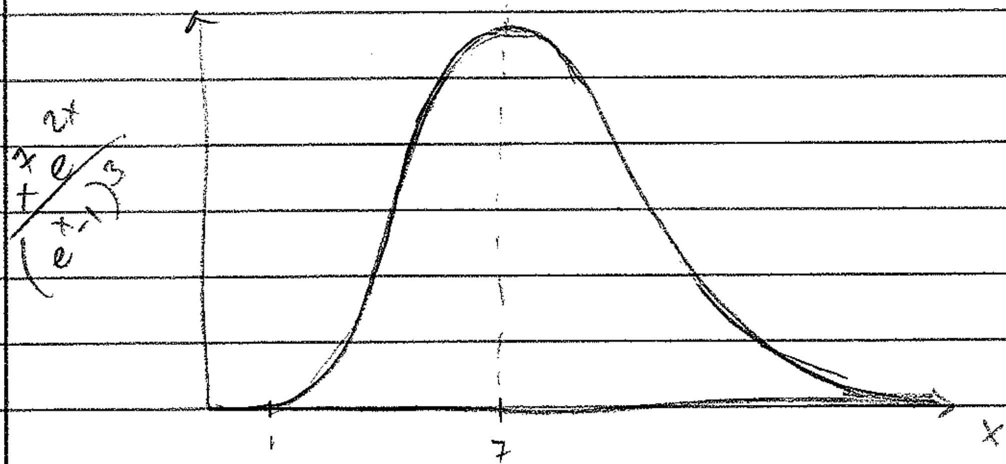
(iii). $\rho\kappa_\nu = \underbrace{1.32 \times 10^{56}}_{A} \frac{\rho^2 g_{ff}}{\nu^3 T^{1/2}} \left(1 - \exp\left(\frac{-h\nu}{kT}\right)\right)$

$$= A \frac{\rho^2 g_{ff} h^3}{k^3 T^{7/2}} \frac{(1 - e^{-x})}{x^3}$$

$$\Rightarrow \frac{1}{\rho\kappa_\nu} \frac{\partial B_\nu}{\partial T} = \frac{2k^3T^2}{h^2c^2} \frac{k^3 T^{7/2}}{A\rho^2 g_{ff} h^3} \frac{x^3}{(1 - e^{-x})} \frac{x^4 e^x}{(e^x - 1)^2}$$

$$= \frac{2k^6 T^{11/2}}{A\rho^2 g_{ff} h^5 c^2} \frac{x^7 e^x}{(e^x - 1)^2 \left(\frac{e^x - 1}{e^x}\right)}$$

3. (iii) (cont.) $\boxed{\dfrac{1}{\rho\chi_\nu}\dfrac{\partial B_\nu}{\partial T} = \dfrac{2k^6T^{11/2}}{A\rho^2 g_{ff}h^5c^2}\dfrac{x^7e^{2x}}{(e^x-1)^3}}$



The largest contribution is from $x \simeq 7$, or $\nu \sim \dfrac{7kT}{h}$

(iv). Rosseland mean opacity is defined as:

$$\frac{1}{\kappa} = \frac{\int \frac{1}{\chi_\nu}\frac{dB_\nu}{dT}\,d\nu}{\int \frac{dB_\nu}{dT}\,d\nu} \propto \frac{T^{11/2}\rho^{-1}}{T^2}\frac{\int \frac{x^7e^{2x}}{(e^x-1)^3}\,d\nu}{\int \frac{x^4e^x}{(e^x-1)^2}\,d\nu}$$

$$\frac{1}{\kappa} \propto \frac{T^{13/2}\rho^{-1}}{T^3}\frac{\int \frac{x^7e^{2x}}{(e^x-1)^3}\,dx}{\int \frac{x^4e^x}{(e^x-1)^2}\,dx}$$

$$\frac{1}{\kappa} \propto \rho^{-1}T^{7/2}$$

$$\Rightarrow \boxed{\kappa \propto \rho T^{-7/2}}$$

Problem 4

Energy is generated by the CNO cycle, so $\varepsilon = \varepsilon_0 \rho T^{17}$

opacity is due to Thompson scattering, so $\kappa =$ constant

define:

$$r = M^a \bar{r}(m)$$
$$\rho = M^b \bar{\rho}(m)$$
$$L = L^c \bar{L}(m)$$
$$T = T^d \bar{T}(m)$$
$$P = P^e \bar{P}(m)$$

as in class

use these scaling relations in the equations of stellar structure

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$
$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho}$$
$$\frac{dL}{dM} = \varepsilon_0 \rho T^{17}$$
$$\frac{dT}{dM} = \frac{-3\kappa L}{64\pi^2 a r^4 T^3}$$

balancing the resulting exponents of M gives:

$$e - 1 = 1 - 4a$$
$$a - 1 = -2a - b$$
$$c - 1 = b + 17d$$
$$d - 1 = c - 4a - 3d$$

to get a fifth equation, need to use the equation of state

the problem does not specify if pressure should be dominated by gas (intermediate to high mass stars) or radiation (very high mass stars), so either case can be assumed

Gas Pressure:

$$P = \frac{\rho K T}{\mu m_H}$$

balancing exponents gives

$$e = b + d$$

this case has a solution

$a = 4/5$
$b = -7/5$
$c = 3$
$d = 1/5$
$e = -6/5$

from the earlier definitions,

$R \propto M^a$
$L \propto M^c$

so $R \propto M^{4/5}$
$L \propto M^3$

using the formula from class, $L \propto T_{eff}^{\frac{4c}{c-2a}} \propto T_{eff}^{8.6}$

This corresponds to the upper main sequence

Problem 5

Start with the pressure integral:

$$P = \frac{1}{3}\int_0^{p_F} p\, v\, n(p)\, dp$$

where $n(p)\,dp = 2 \cdot \frac{4\pi p^2}{h^3}\,dp$

for highly relativistic electrons, $v \sim c$

so $$P = \frac{8\pi c}{3h^3}\int_0^{p_F} p^3\, dp = \frac{2\pi c}{3h^3}\, p_F^4$$

using $n_e = \int_0^{p_F} n(p)\,dp = \int_0^{p_F} \frac{8\pi p^2}{h^3}\,dp$

gives $$n_e = \frac{8\pi}{3h^3}\, p_F^3$$

or $$p_F = \left(\frac{3h^3}{8\pi}\right)^{1/3} n_e^{1/3}$$

so $$P = \frac{2\pi c}{3h^3}\left(\frac{3h^3}{8\pi}\right)^{4/3} n_e^{4/3}$$

$$P = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} n_e^{4/3}$$

$$n_e = \frac{\#\text{ electrons}}{\text{Volume}} = \frac{\#\text{ protons}}{\text{Volume}} = \frac{\#\text{ protons}}{\#\text{ baryons}}\;\frac{\#\text{ baryons}}{\text{Volume}}$$

$$= \frac{\#\text{ protons}}{\#\text{ baryons}}\;\frac{\text{mass of star}}{\text{mass of baryon}}\;\frac{1}{\text{volume}}$$

$$= \frac{Z}{A}\frac{\rho}{m_p}$$

so, finally, $$P = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{Z}{A m_p}\right)^{4/3}\rho^{4/3}$$

or $P \propto \rho^{4/3}$

therefore, this equation of state is a polytrope with $\gamma = 4/3$

$\gamma = \frac{n+1}{n}$, so $n=3$ if $\gamma = 4/3$

From class,

$$M = 4\pi\alpha^3 \rho_c \left[-\xi^2 \frac{d\theta}{d\xi}\right]_{\xi_1}$$

where $\alpha = \left[\frac{n+1}{4\pi G} K \lambda^{1/n - 1}\right]^{1/2}$

for a polytrope of index n

for $n=3$, $\alpha = \left[\frac{K}{\pi G}\right]^{1/2} \lambda^{-1/3}$

$\lambda \equiv \rho_c$, so

$$M = 4\pi \left[\frac{K}{\pi G}\right]^{3/2} \rho_c^{-1} \rho_c \left[-\xi^2 \frac{d\theta}{d\xi}\right]_{\xi_1}$$

ρ_c cancels, and K is given by

$$P = K\rho^{4/3} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{Z}{Am_p}\right)^{4/3}\rho^{4/3}$$

therefore, $K = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{Z}{Am_p}\right)^{4/3}$

we are told to assume the white dwarf has no H, which means it is entirely composed of some combination of He, C, and O, which all have $\frac{Z}{A} = 2$

using $\frac{Z}{A} = 2$, and looking up

$\left[-\xi^2 \frac{d\theta}{d\xi}\right]_{\xi_1}$ in a table for $n=3$

gives $M = 1.44\ M_\odot$