Ay 123 Pset 5 Solutions

1. a) $\frac{dy}{dx} = f(x)$

In Euler's scheme,

$$y(x_0+h) = y(x_0) + h\left.\frac{dy}{dx}\right|_{x_0} = y(x_0) + hf(x_0)$$

An exact expression for y is given by Taylor Expansion:

$$y(x_0+h) = y(x_0) + hf(x_0) + \frac{1}{2}h^2 f'(x_0) + \frac{1}{6}h^3 f''(x_0) + \ldots$$

the largest term (for $x<1$) in the difference between these expressions is

$$\frac{1}{2}h^2 f'(x_0)$$

so the error goes as h^2

b) In the midpoint technique, f is evaluated in the middle of the step interval, not at the end.

$$y(x_0+h) = y(x_0) + hf(x_0 + h/2)$$

the second term can be Taylor expanded:

$$hf(x_0+h/2) = hf(x_0) + \frac{h^2}{2}f'(x_0) + \frac{1}{2}\frac{h^3}{4}f''(x_0) + \ldots$$

so $y(x_0+h) = y(x_0) + hf(x_0) + \frac{h^2}{2}f'(x_0) + \frac{h^3}{8}f''(x_0) + \ldots$

The difference between this expression and the Taylor expansion is the h^3 term, so the error goes as h^3.

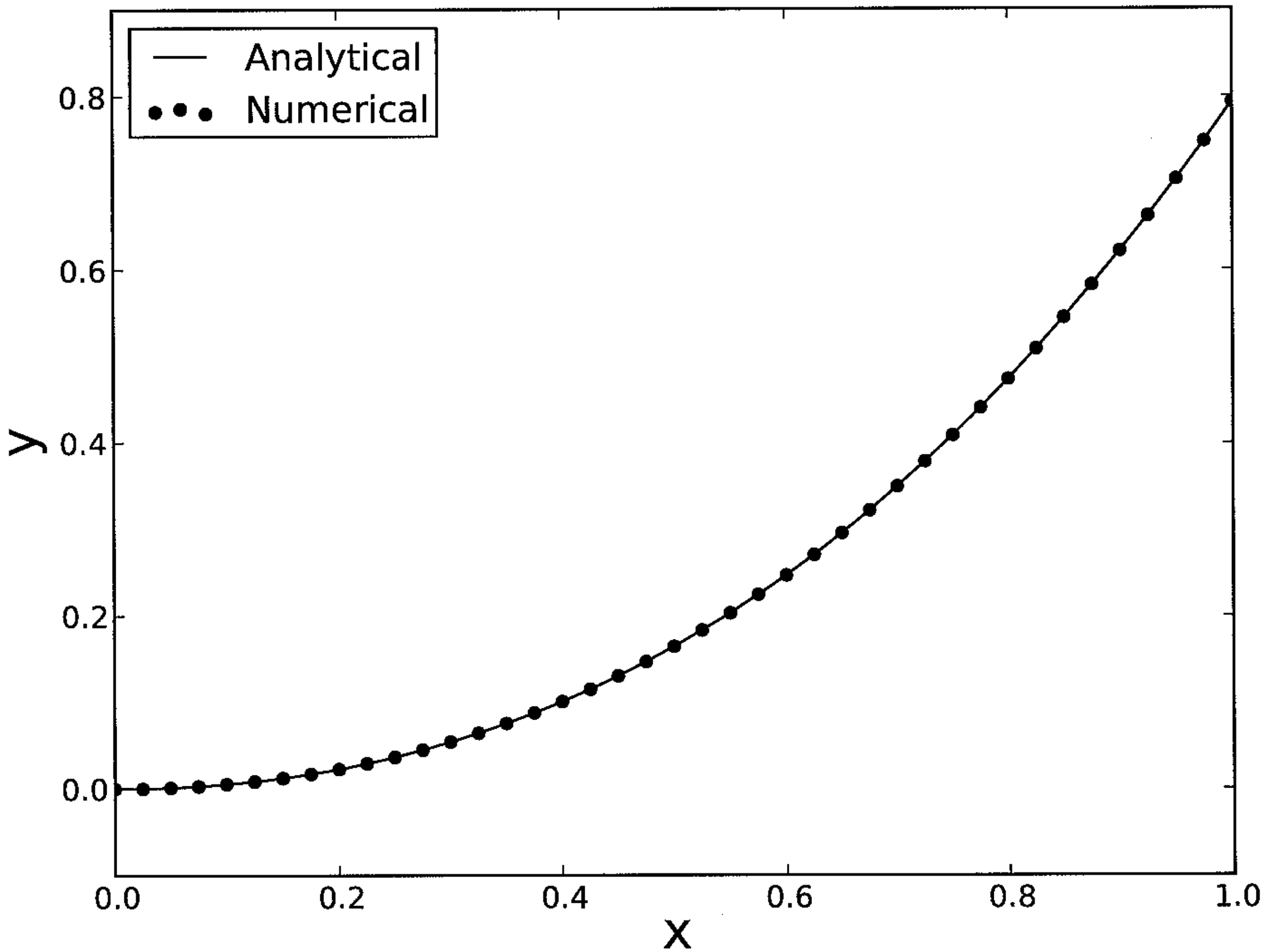
c) $\frac{dy}{dx} = x^2 + \sin x$

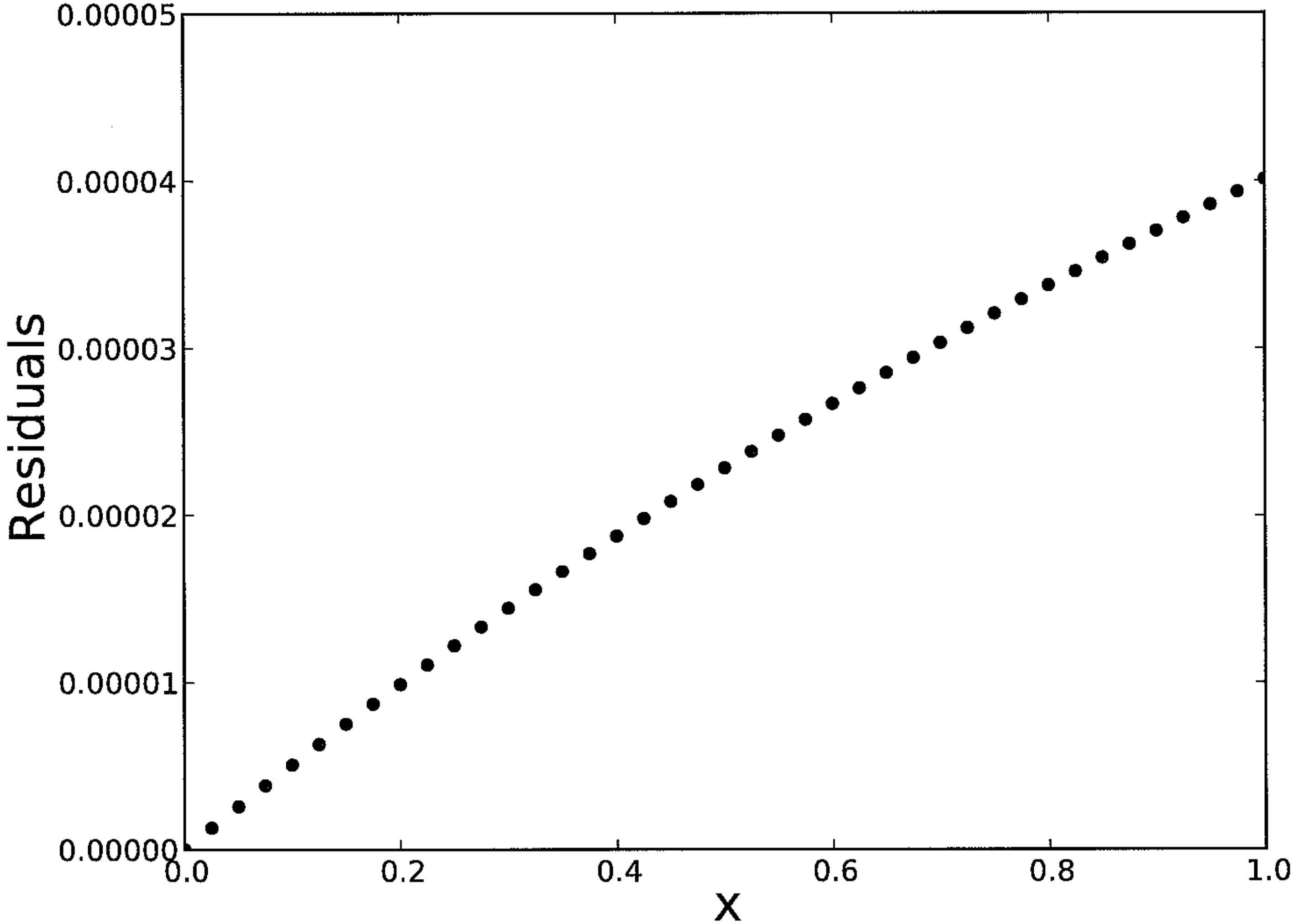
integrating gives $y(x) = \frac{x^3}{3} - \cos x + C$

choosing $y(0) = 0$ gives $C = 1$

d) see plot

the largest absolute error is 4×10^{-5} at $x=1$
the largest fractional error is 4×10^{-3} at $x = 0.025$, since y is small there


Analytical
Numerical
y
x
0.0
0.2
0.4
0.6
0.8
1.0



Residuals
0.00005
0.00004
0.00003
0.00002
0.00001
0.00000
0.0
0.2
0.4
0.6
0.8
1.0
x

2\. a) For simplicity in this problem, I am writing $\frac{M}{M_\odot}$ as M, $\frac{R}{R_\odot}$ as R, $\frac{L}{L_\odot}$ as L, and $\frac{T_{eff}}{T_{eff\odot}}$ as T

The equation defining the Hayashi Track is $\log L = 15 \log T + 0.2 \log M + \delta$ (1)

Determine δ using the Sun, which has $L = 40$ at the start of the track:

$$R_0 = \frac{43.2}{1-0.2X} M \qquad (2)$$

$X = 0.9$, and $M = 1$ for the Sun, so $R_0 = 52.68$

Rewrite T in terms of L and R:

$$L = R^2 T^4$$

$$\log T = \frac{\log L}{4} - \frac{\log R}{2} \qquad (3)$$

Plug this into equation 1, and solve for δ to find $\delta = 8.51$

So the Hayashi track can be plotted on an HR diagram as

Hayashi Track $$\boxed{\log L = 15 \log T + 0.2 \log M + 8.51}$$

The start of the Hayashi track can be found for each mass using equation 2 to find R_0. Then use equations 1 and 3 to solve for T in terms of R and M:

$$\log T = \frac{2 \log R - 0.2 \log M - \delta}{11}$$

The end of the Hayashi track is given by intersection with the Henyey track.

The Henyey track is given by

$$L \propto M^{5.5} R^{-0.5}$$

so $\log L = 5.5 \log M - 0.5 \log R + \gamma$ (4)
where γ is a constant

γ can be found using the fact that the Henyey track ends on the main sequence:

$$L \propto M^3 \quad R \propto M^{(n-1)/(n+3)}$$

for $\varepsilon \propto \rho T^n$

For the lower main sequence, these proportionalities are equalities (to match the Sun).
Using $n = 4$ gives

$$L = M^3, \quad R_{Lower} = M^{3/7}$$

plugging these into equation 4:

$$3 \log M = 5.5 \log M - \frac{3}{14} \log M + \gamma$$

$$\gamma_{Low} = -\frac{16}{7} \log M \quad \text{for } M < 2$$

The main sequence must be continuous at $2 M_\odot$, so for the upper main sequence, $L = M^3$ must continue.
Matching R at $M = 2$ (for $n = 16$ on the upper main sequence) gives

$$K + \frac{15}{19} \log M = \frac{3}{7} \log M \quad \text{at } M = 2$$

$$K = -\frac{48}{133} \log 2$$

Therefore, $\log R_{upper} = \frac{15}{19} \log M - \frac{48}{133} \log 2$, and

$$3 \log M = 5.5 \log M - \frac{15}{38} \log M + \frac{48}{266} \log 2 + \gamma_{up}$$

$$\gamma_{up} = -\frac{40}{19} \log M - \frac{48}{266} \log 2 \quad \text{for } M > 2$$

note that $\gamma_{low} = \gamma_{up}$ at $M = 2$

Putting this all together results in the Hayashi track being given by:

$$\log L = 5.5 \log M - 0.5 \log R - \frac{16}{7} \log M, \quad M < 2$$
$$\log L = 5.5 \log M - 0.5 \log R + \frac{40}{19} \log M + \frac{48}{266} \log 2, \quad M > 2$$

Next, use equation 3 to substitute for $\log R$:

Henyey Track

$$\boxed{\begin{aligned} \log L &= \frac{30}{7} \log M - \frac{4}{3} \log T & M < 2 \\ \log L &= \frac{578}{57} \log M - \frac{4}{3} \log T - \frac{32}{133} \log 2 & M > 2 \end{aligned}}$$

equating these expressions to equation 1 gives the intersection points:

$M/M_\odot$	$R/R_\odot$	$L/L_\odot$	T (K)
0.5	4.49	0.051	1295
1	9.10	0.332	1453
2	18.4	2.16	1632
4	38.8	15.7	1845
8	81.7	114	2086
16	172	823	2359

The Henyey track runs from those points to the main sequence. The temperature at the main sequence can be found using $L \propto M^3$, $R \propto M^{(n-1)/(n+3)}$ with equation 3:

$$\log T = \frac{3}{4}\log M - \frac{3}{14}\log M = \frac{15}{28}\log M \quad \text{for } M<2$$

$$\log T = \frac{3}{4}\log M - \frac{15}{38}\log M + \frac{24}{133}\log 2$$

$$\log T = \frac{27}{76}\log M + \frac{24}{133}\log 2 \quad \text{for } M>2$$

So, in summary, the Hayashi track is given by:

$$\log L = 15\log T + 0.2\log M + 8.51$$

and starts at

$$\log T = \frac{2}{11}\log R_0 - \frac{1}{55}\log M - 8.51$$

with $R_0 = 52.68\,M$

The track ends at the points given in the table.

The Henyey track is given by

$$\log L = \frac{30}{7}\log M - \frac{4}{3}\log T \qquad M<2$$

$$\log L = \frac{578}{57}\log M - \frac{4}{3}\log T - \frac{32}{133}\log 2 \qquad M>2$$

It starts at the table points and ends at:

$$\log T = \frac{15}{28}\log M \qquad M<2$$

$$\log T = \frac{27}{76}\log M + \frac{24}{133}\log 2 \qquad M>2$$

The main sequence is given

$\log L = 3/\log M$ for all M

$\log T = \frac{15}{28} \log M \quad M < 2$

$\log T = \frac{27}{76} \log M + \frac{24}{133} \log 2 \quad M > 2$

see attatched plot for tracks

b) Luminosity is powered by collapse.

$$L = -\frac{1}{2}\frac{d\Omega}{dt} = -\frac{1}{5}\frac{GM^2}{R^2}\frac{dR}{dt}$$

note - M and R are now in cgs units

First, convert this equation to solar units and time to years.

$$\frac{L}{L_\odot} = -\frac{1}{5}\frac{(M/M_\odot)^2}{(R/R_\odot)^2}\frac{d(R/R_\odot)}{d(t/yr)}\frac{GM_\odot^2}{R_\odot L_\odot (1\,yr)}$$

$$\frac{L}{L_\odot} = -6.26 \times 10^6 \frac{(M/M_\odot)^2}{(R/R_\odot)^2}\frac{d(R/R_\odot)}{d(t/yr)}$$

Now switching to the notation of part a, with $M = \frac{M}{M_\odot}$, etc. and t in years

$$L = -6.26 \times 10^6 \frac{M^2}{R^2}\frac{dR}{dt}$$

On the Hayashi track,

$$L = T^{15} M^{1/5} 10^{6}$$

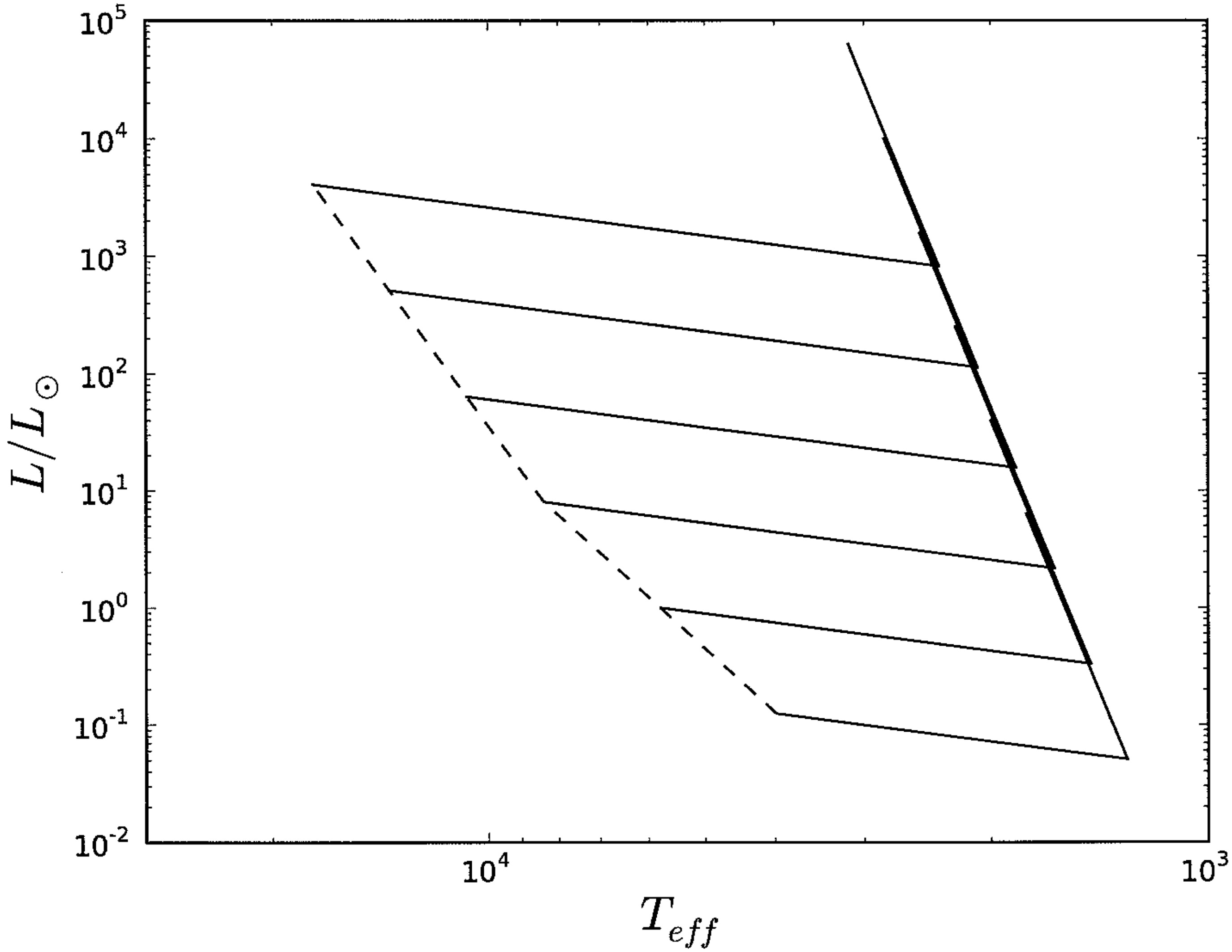
using equation 3 to replace T and solving for L gives

$$L = R^{30/11} M^{-4/55} 10^{-48/11}$$

so $$-6.26 \times 10^6 \frac{M^2}{R^2}\frac{dR}{dt} = R^{30/11} M^{-4/55} 10^{-48/11}$$

integrating this gives

$$\Delta t_{Hay} = 2.09 \times 10^9 M^{114/55}\left(R_{int}^{-41/11} - R_0^{-41/11}\right)$$


10^5
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
$L/L_\odot$
10^4
10^3
T_{eff}

On the Henyey track, $L = M^{5.5} R^{-0.5} 10^{\gamma}$

so $\frac{dR}{dt} = -\frac{1}{6.26 \times 10^{6}} M^{-3.5} R^{1.5} 10^{\gamma}$

and

$$\Delta t_{Hen} = 1.25 \times 10^{7-\gamma} M^{-3.5} \left(R_{MS}^{-0.5} - R_{int}^{-0.5} \right)$$

using γ, R_0, R_{MS}, and R_{int} from part a) gives (in years)

$M/M_\odot$	t_{Hay}	t_{Hen}	t_{total}
0.5	1.84×10^{6}	2.00×10^{7}	2.18×10^{7}
1	5.56×10^{5}	8.36×10^{6}	8.91×10^{6}
2	1.70×10^{5}	3.39×10^{6}	3.56×10^{6}
4	4.42×10^{4}	1.01×10^{6}	1.06×10^{6}
8	1.16×10^{4}	3.02×10^{5}	3.34×10^{5}
16	3.04×10^{3}	8.98×10^{4}	9.29×10^{4}

3. Gravitational energy released by the collapsing protostar:

$$U = \frac{GM^2}{R_c}$$, where R_c is the radius at which the temp. exceeds T_c.

Assuming all this gravitational energy goes to thermal energy:

$$E_{th} \doteq NKT = \frac{M}{m_H} KT_c = \frac{GM^2}{R_c}$$

Assuming approximately uniform density:

$$\rho = \frac{M}{\frac{4\pi}{3} R^3}$$

But we know $R_c = \frac{GMm_H}{KT_c}$

$$\Rightarrow \rho_c = \frac{3M}{4\pi}\left(\frac{KT_c}{GMm_H}\right)^3 = \boxed{\frac{3}{4\pi M^2}\left(\frac{KT_c}{Gm_H}\right)^3 = \rho_c}$$

A gas becomes degenerate when the Fermi energy becomes comparable to the thermal energy of the e^-'s.

$$E_F = \frac{h^2}{2m_e}\left(\frac{3n_e}{8\pi}\right)^{2/3} \approx KT$$

Since $n_e = \frac{\rho}{\mu_e m_H}$

3. (cont.) $\rho_{crit} = \frac{8\pi}{3} \mu_e m_H \left(\frac{2 m_e k T}{h^2} \right)^{3/2}$

This critical density is the density at which a gas becomes degenerate.

If a star is halted by degeneracy pressure and at the onset of fusion, $\rho_{crit}(T_c) \stackrel{\Delta}{=} \rho_c$.

$$\frac{8\pi}{3} \mu_e m_H \left(\frac{2 m_e k T_c}{h^2} \right)^{3/2} \leq \frac{3}{4\pi M^2} \left(\frac{k T_c}{m_H G} \right)^3$$

$$M^2 \leq \frac{9}{32\pi^2 2^{3/2}} \frac{h^3 c^3}{G^3 \mu_e m_H^4} \left(\frac{k T_c}{m_e c^2} \right)^{3/2}$$

$$M_{crit} = \frac{3}{8\pi 2^{1/4}} \frac{h^{3/2} c^{3/2}}{G^{3/2} \mu_e^{1/2} m_H^2} \left(\frac{k T_c}{m_e c^2} \right)^{3/4}$$

Given that the Chandrasekhar mass for an object is:

$$M_{ch} = \left(\frac{9}{32\pi^2} \frac{1}{\mu_e m_H} \right)^{1/2} \left(\frac{hc}{\mu_e m_H} \frac{5}{3G} \right)^{3/2}$$

$$\Rightarrow \boxed{\frac{M_{crit}}{M_{ch}} = \mu_e^{3/2} \frac{3^{3/2}}{2^{3/4}} \frac{1}{\pi 5^{3/2}} \left(\frac{k T_c}{m_e c^2} \right)^{3/4}}$$