1 Basic Fluid Mechanics

Before we dive into the basic equations of fluid mechanics let us review vector algebra (from Physics 1). Consider a vector field, \( \mathbf{E} \). We are interested in computing the integral flux of this field emanating from a region \((x, y)\) (where for simplicity we have reverted to two dimensions). The flux for each “face” is the dot product of the field with the normal multiplied by the “area” of that face. The normal is always facing “outward”. Then net flux across the two faces in the \( x \) direction is \( \frac{\partial \mathbf{E}}{\partial x} \delta x \delta y \) and a similar expression holds for the pair of \( y \) faces. Note that for the two dimensions under consideration \( \delta x \delta y \) is the volume of the element, \( \delta V \). Thus the net flux through the cube of \( \delta x, \delta y, \delta z \) is \( \frac{\partial \mathbf{E}}{\partial x} \delta x \delta y + \frac{\partial \mathbf{E}}{\partial y} \delta y \delta z \). Switch to three dimensions the net flux is \( \nabla \cdot \mathbf{E} \delta V \). We have thus arrived at Gauss’ law.

Consider a fluid element with volume \( \Omega \) (which is a small volume). At position \( \mathbf{r} \) and time \( t \) the element has density \( \rho(\mathbf{r}, t) \) and velocity \( \mathbf{v}(\mathbf{r}, t) \). The mass contained within this volume is \( \rho(\mathbf{r}, t) \Omega \). The flux of fluid matter across a closed surface, \( d\mathbf{S} \), is \( \int d\mathbf{S} \cdot \rho \mathbf{v} \) which, using Gauss’ law is \( \nabla \cdot (\rho \mathbf{v}) \Omega \). Over time \( \delta t \) the loss is \( \nabla \cdot (\rho \mathbf{v}) \Omega \delta t \) of matter which must be balanced by \( \delta (\rho \Omega) \) which is equal to \( \Omega \frac{\partial \rho}{\partial t} \delta t \) since, by construction, \( \Omega \) is fixed. We thus derive the first equation of fluid mechanics – the conservation of mass:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\] (1)

In order to derive the second equation – the equation of force (or momentum balance) – we need to understand “convective” derivative. \( \rho(\mathbf{r}, t) \) and \( \mathbf{v}(\mathbf{r}, t) \) are simply coordinate based vector field description of the fluid density and velocity. However, the forces act on matter and we need to compute the acceleration of each particle and track its evolution under internal (fluid pressure) and external forces (e.g. gravity). Consider some quantity, \( f \) of the fluid at \( (\mathbf{r}, t) \). In time \( t + \delta t \) the fluid element \( (\mathbf{S} \text{ and } \Omega) \) would have moved to position \( \mathbf{r} + \mathbf{v} \delta t \). Thus \( \delta f \) at time \( t + \delta t \) has the approximate value \( \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f \delta t \). This leads to the convective derivative

\[
\frac{D}{Dt} f \equiv \left[ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] f
\] (2)
The definition can be extended to vector quantities (e.g. \( \mathbf{f} \)) by treating each component of the vector quantity as a scalar [e.g. \( \mathbf{f} = (f_x, f_y, f_y) \)].

We can now write Newton’s second law for fluids:

\[
\Omega \rho \frac{D \mathbf{v}}{Dt} = \mathbf{F}_{\text{pressure}} + \mathbf{F}_{\text{gravity}}. \quad (3)
\]

This is really three equations (for each of the three directions: \( \hat{x} \), \( \hat{y} \) and \( \hat{z} \)) but written compactly. Here, we ignore “viscous” forces (forces which act when two fluids of different velocities interact) since in astrophysical conditions viscosity is important only in “shock fronts” (and there we use the artifice of jump conditions to avoid going into the details of viscous forces).

Following Draine (p. 390), let the surface element \( dS \) be a vector in the outward direction from the closed surface \( S \). The pressure of the external fluid pushes inward at each point on the surface and the net force on the fluid element is

\[
\mathbf{F}_{\text{pressure}} = \int (-p) dS = \int -\nabla p \Omega \rightarrow -\Omega \nabla p. \quad (4)
\]

Above we converted a surface integral to a volume integral using Gauss’ theorem (as before). The magnetic pressure acts the same, replacing \( p \) by \( \frac{B^2}{8\pi} \).

The force due to gravity is mass times acceleration due to gravity or \( \rho \Omega (-\nabla \phi) \) where \( \phi \) is the gravitational potential. Thus we obtain

\[
\rho \frac{D \mathbf{v}}{Dt} = -\nabla \left( p + \frac{B^2}{8\pi} \right) - \rho \nabla \phi. \quad (5)
\]

Nominally, the three principal parameters of a simple fluid, \( P \), \( T \) and \( \rho \) are independent. For an ideal fluid these three quantities are related by Boyle’s law. However, if additional conditions are applicable then the fluid can be described by only of the three quantities. For example, if all changes are isothermal then \( P \propto \rho \). On the other hand, if the changes are too rapid to have any heat transfer (adiabatic) then \( P = K \rho^\gamma \) where \( \gamma \) is the adiabatic exponent and is equal to 5/3 for simple gas. This simplification allows us to bypass the equation for conservation of energy (which tends to be long and messy). Fortunately, in most astronomical sites, the adiabatic assumption holds very well.

### 2 Acoustic Waves

In one dimension our two basic equations (Equations 1 and 5), ignoring gravitational force, simplify to:

\[
\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0 \quad (6)
\]
\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial x} = 0. \quad (7)
\]
Let us assume that initially our fluid is uniformly distributed, density $\rho_0$, and is in equilibrium, $P_0$, throughout its volume. We are at rest with the fluid, $v = 0$. We introduce induce a velocity, $v$, to a specific fluid element. This velocity is smaller than the sound speed. This disturbance will propagate inducing change $p_1(r, t) \ll P_0$ and $\rho_1(r, t) \ll \rho_0$. Keeping only first order terms in our equation of continuity (Equation 6 and equation of force (Equation 7) we obtain

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v_1}{\partial x} = 0 \quad (8)$$

$$\frac{\partial v_1}{\partial t} + \frac{1}{\rho_0} \frac{\partial p_1}{\partial x} = 0 \quad (9)$$

Differentiate the first equation with respect to $t$ and the second equation with respect to $x$ and then combining appropriately we obtain:

$$\frac{\partial^2 \rho_1}{\partial t^2} - \frac{\partial^2 p_1}{\partial x^2} = 0 \quad (10)$$

We assert that these small disturbances are adiabatic. If so,

$$\delta P = K \gamma \rho_0^{\gamma-1} \delta \rho \quad (11)$$

$$p_1 = \gamma \left( \frac{P_0}{\rho_0} \right) \rho_1 = a_0^2 \rho_1$$

where $a_0^2 = \gamma P_0/\rho_0$. Substituting Equation 11 into Equation 10 we obtain

$$\frac{\partial^2 \rho_1}{\partial t^2} - a_0^2 \frac{\partial^2 \rho_1}{\partial x^2} = 0. \quad (12)$$

Finally, we have an Equation so familiar to students of Ph1 – the wave equation! The solution to this equation is $\rho_1 \propto \exp(i\omega t + ikx)$ and the relation between $\omega$ and $k$ is given by the “dispersion” relation:

$$\omega^2 = a_0^2 k^2. \quad (13)$$

In the above equation, we identify

$$a_0 = \left( \frac{\gamma k_B T}{\mu m_H} \right)^{1/2} \quad (14)$$

as the ("adiabatic") speed of sound waves. However, there is HUGE difference between the corresponding equation we derived for electromagnetic waves and acoustic waves. For the former, the speed of electromagnetic waves is an absolute constant (so puzzling for all elementary and even jaded teachers of Special Relativity) whereas the acoustic speed is
simply defined by Equation 14. In particular, it depends on the specific ratio of pressure over density (which is not a Universal constant). Recognizing this basic difference we note:

\[(a_0 + a')^2 = a_0^2 \left( \frac{\rho_0 + \rho_1}{\rho_0} \right)^{\gamma - 1}
\]

\[a' = \frac{1}{2(\gamma - 1)} \rho' \rho_0.
\]

(15)

Thus the sound speed increases with pressure provided \(\gamma > 1\). Indeed, this is where the importance of “adiabatic” perturbations become important. For almost all cases under consideration (e.g. terrestrial, astronomical) fluids, when perturbed, are not isothermal but adiabatic (that is, there is little exchange of energy, on relevant timescales). Alas, this means that fluid mechanics is fundamentally non-linear (unlike electromagnetic phenomenon).

2.1 Steepening of Acoustic Waves: Shocks

You can convince yourself that any perturbation in which say the density (or equivalently, pressure) increases gradually (before decreasing) will steepen. This steepening results in the formation of a discontinuity – a shock wave. For ordinary acoustic waves (e.g. when you speak) the steepening results in dissipation of energy (imagine what happens when the lower density perturbation arrives and collides with the higher density perturbations). In other words, we have not included dissipation (viscosity) in the above equation and if we did include then acoustic waves die down. In contrast, EM waves simply decrease as \(r^{-1}\) (for the amplitude), unless they encounter a medium which has some resistivity.

In astronomical locales, the bulk speed of ejecta (nova, supernova), winds (from stars), heated material (e.g. HII regions), colliding galaxies is “super-sonic” (perhaps a better term is “hyper”-sonic) that strong shock fronts are readily formed.