NOZZLES

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1. Motivation

In astronomy textbooks it is usually stated that the Bondi-Hoyle solution has a specific value for the accretion rate (given the boundary conditions of ambient density and temperature of the gas). Only with this specific accretion rate will there be a seamless transition from sub-sonic inflow to super-sonic inflow.

The Bondi-Hoyle or the Parker problem is quite analogous to the rocket problem, in that both involve a smooth transition from sub-sonic to super-sonic flow. What bothered me is the statement in astronomy books that the solution demands a specific accretion (or excretion) mass rate. In contrast, the thrust on a rocket can be varied (as can be gathered if you watch NASA TV). Given the similarity between the astronomical problem and the rocket problem I thought an investigation of the latter may be illuminating and hence this note.

2. The de Laval Nozzle

Consider a rocket which is burning fuel. Let $\dot{M}$ be the burn rate of the fuel and u_e be the speed of the exhaust. Then in steady state the vertical "thrust" or force is $\dot{M}u_e$. The resulting vertical thrust lifts the rocket. Clearly, it is of greatest advantage to make the exhaust speed as large as possible and to minimize the outflow in directions other than vertical.

A rocket engine consists of a chamber in which fuel is burnt connected to a nozzle. A properly designed[1] nozzle can convert the hot burnt fuel into supersonic flow (see Figure 1).

The integration of the momentum equation (under assumptions of inviscid flow) yields the much celebrated Beronulli's theorem. Specifically, the following sum

$$\mathcal{B} = \frac{1}{2}|u|^2 + \phi + \int \frac{dP}{\rho} \tag{1}$$

is constant along a stream line (with $\mathcal{B}$ being the streamline constant). Here ϕ is a scalar potential associated with the body force (and for terrestrial and astronomical cases it is the gravitational field).

[1]R. Goddard was the first to apply the de Laval nozzle to rocketry.

Let $A(x)$ be the cross-sectional area of the nozzle. Assuming steady state, the equation of continuity and Bernoulli's equation are

$$\rho(x)u(x)A(x) = \text{constant} \tag{2}$$

$$\frac{1}{2}u^2 + \int \frac{dP}{\rho} = \text{constant}; \tag{3}$$

where we have ignored the gravitational potential because (for rockets) the pressure gradients are much stronger than acceleration due to gravity. Henceforth we will drop the explicit dependence of ρ, u and A with x.

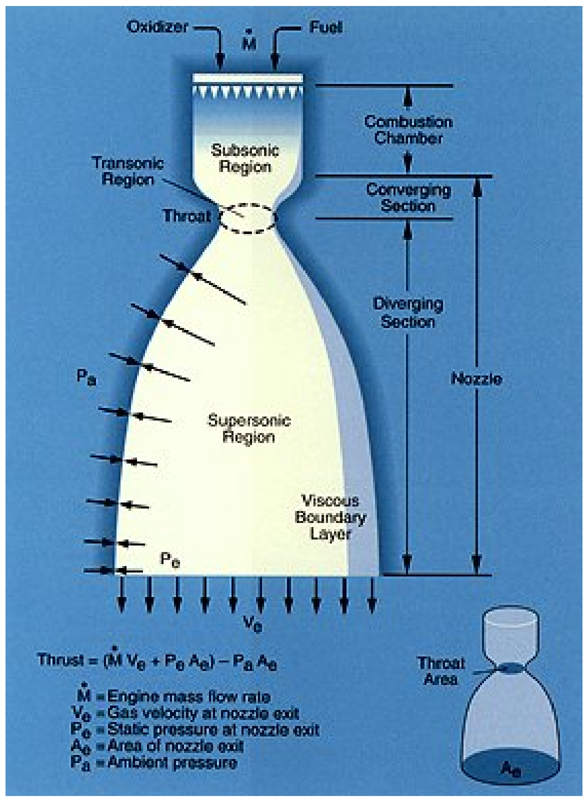

FIGURE 1. A typical nozzle and associated terminology.

Differentiating Equation 3 we find

$$\begin{aligned} udu + \frac{1}{\rho}\frac{dP}{d\rho}d\rho &= 0 \\ \frac{d\rho}{\rho} &= -\mathcal{M}^2\frac{du}{u} \end{aligned} \tag{4}$$

where $c_s^2 = dP/d\rho$ is the square of the sound speed and $\mathcal{M} = u/c_s$ is the Mach number. We immediately see that for highly sub-sonic flow ($\mathcal{M} \ll 1$), the fluid is almost incompressible.

Next, we take the log derivative (i.e. take the log and then differentiate) of Equation 3 and find

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0. \tag{5}$$

Combining Equations 4 and 5 we obtain

$$(1 - \mathcal{M}^2)\frac{du}{u} = -\frac{dA}{A}. \tag{6}$$

From Equation 6 we see that for sub-sonic flows ($\mathcal{M} \leq 1$) du/u has the opposite sign of dA/A. Thus the flow speeds up when the nozzle starts constricting and slows down when the nozzle expands. In contrast, for supersonic flows ($\mathcal{M} \geq 1$) we find du/u and dA/A enjoy the same sign. Thus the fluid speeds up as the cross-section of the nozzle increases!

The "throat" of a nozzle is the narrowest cross-section ($x = x_t$). Thus $dA/dx = 0$ at $x = x_t$ and accordingly

$$(1 - \mathcal{M}^2)\frac{du}{u}\bigg|_{x=x_t} = 0 \tag{7}$$

There are two possibilities: either $du/u(x_t) = 0$ or $\mathcal{M}(x_t) = 1$.

If the engine is on low power then the flow will be sub-sonic everywhere reaching a maximum value at x_t. [This discussion also shows that a time independent entirely sub-sonice flow is possible. As noted by F. Shu this solution is rarely of interest in the astronomical context.]

We now turn up the burn rate and the flow speed increases at the throat. At a particular burn rate, the flow becomes supersonic with $\mathcal{M}(x_t) = 1$. This burn rate is called as the critical burn rate.

What happens if we still increase the burn-rate? In this case, the flow can be expected to become trans-sonic for $x' < x_t$. However, since $dA/A|_{x_t} \neq 0$ we see from Equation 6 that

$$\frac{du}{u}\bigg|_{x\to x'} \to -\infty. \tag{8}$$

A negative steep gradient in u means that the flow stagnates at x'. The burning of the fuel continues and pressure builds up in the combustion chamber. This build-up moves the stagnation point towards the throat. Once the stagnation point crosses the throat a new steady flow is established. *Thus contrary to the statements made in astronomy books a rocket engine does not work for a fixed $\dot{M}$.* Rockets work for $\dot{M} > \dot{M}_{\rm crit}$ [3]. See below for a description of the Space Shuttle Main Engine.

The sort of adjustment described above takes place because the flow is sub-sonic in the combustion chamber. In this chamber, the flow can rearrange to boundary conditions. In particular, even if c_0 is fixed (related to the temperature of the burning), ρ_0 is determined by the burn rate. In steady state, the burn rate must be matched by $\dot{M}$.

3. Polytropic Gas

Following [4] we work out a toy model for a polytropic fuel, $P = K\rho^\gamma$. The primary variables that we wish to deal with is the density, $\rho_0 = \rho(x=0)$, and the speed of sound in the combustion chamber (c_0) and the steady state mass flux, $\dot{M}$. To this end we note:

$$c_s^2 = \frac{dP}{d\rho} = \gamma\rho^{\gamma-1}$$
$$= c_0^2\Big(\frac{\rho}{\rho_0}\Big)^{\gamma-1}. \quad (9)$$

Thus the speed of sound elsewhere can be expressed as

$$\frac{\rho}{\rho_0} = \Big(\frac{c_s}{c_0}\Big)^{2/(\gamma-1)}. \quad (10)$$

We re-express the equation of continuity and Bernoulli's theorem in our preferred new parameters:

$$\Big(\frac{c_s}{c_0}\Big)^{2/(\gamma-1)} u = \frac{1}{\rho_0}\Big(\frac{\dot{M}}{A}\Big) \quad (11)$$
$$\frac{1}{2}u^2 + \frac{c_s^2}{\gamma-1} = \frac{c_0^2}{\gamma-1}. \quad (12)$$

We have ignored pressure from the ambient medium (which does make a difference in real life).

Let us assume that the flow has become transonic at the throat, $x = x_t$. Here, $A_t = A(x_t)$, the are of the throat aperture. Then setting $u(x_t)$, the flow speed, to the local speed of sound, $c_t = c_s(x_t)$ in Equation 12 we find

$$\frac{c_t}{c_0} = \Big(\frac{2}{\gamma+1}\Big)^{1/2}. \quad (13)$$

Substituting this value into Equation 11 we obtain

$$\dot{M}_{\rm crit} = \Big(\frac{2}{\gamma+1}\Big)^{(\gamma+1)/(2(\gamma-1))} c_0\rho_0 A_t \quad (14)$$
$$= \beta c_0\rho_0 A_t. \quad (15)$$

where β is the numerical factor which can be deduced from Equation 14. Thus the critical mass flux is essentially the same as that obtained by dimensional considerations.

Let us assume that the mass flux, $\dot{M}$, and the speed of the ejecta, u_e are measured. Then, the density of the exhaust is given by

$$\rho_e = \frac{\dot{M}}{A_e u_e} \quad (16)$$

where A_e is the area of the exhaust aperture.

We now wish to determine the Mach number at the exit aperture. In Equation 12 we set $u = u_e$, $c_e = c_s(x_e)$ and $\mathcal{M}_e = u_e/c_e$. Dividing this equation by c_e we find

$$\frac{1}{2}u_e^2 + \frac{1}{\gamma - 1}c_e^2 = \frac{1}{\gamma - 1}c_0^2 \tag{17}$$

which can be re-expressed as

$$c_0^2 = \frac{\gamma - 1}{2}u_e^2 + c_e^2. \tag{18}$$

Our plan is to eliminate c_0. Starting with Equation 9 and substituting the densities (Equations 15 and 22)

$$\begin{aligned} c_0^2 &= c_e^2\left(\frac{\rho_0}{\rho_e}\right)^{(\gamma-1)} \\ &= c_e^2\left(\frac{\dot{M}/(\beta c_0 A_t)}{\dot{M}/(u_e A_e)}\right)^{(\gamma-1)} \\ &= c_e^2\left(\frac{u_e A_e}{\beta c_0 A_t}\right)^{(\gamma-1)}. \end{aligned} \tag{19}$$

Thus

$$c_0^2 = c_e^{4/(\gamma+1)}\left(\frac{u_e A_e}{\beta A_t}\right)^{2(\gamma-1)/(\gamma+1)}. \tag{20}$$

Eliminating c_0 from Equations 18 and 20 we find

$$\frac{\gamma - 1}{2}u_e^2 + c_e^2 = c_e^{4/(\gamma+1)}\left(\frac{u_e A_e}{\beta A_t}\right)^{2(\gamma-1)/(\gamma+1)}. \tag{21}$$

From this equation, c_e can be estimated through numerical means. The Mach number at the exit aperture is then $M_e = u_e/c_e$.

4. Application to SSEM

We apply this result to the Space Shuttle Main Engine (SSEM; see §A). The Shuttle has three SSEMs. For each, running at 100% throttle the mass flux is $\dot{M} \approx 438\,\mathrm{kg\,s^{-1}}$ and the exhaust velocity is $v_e = 5.18\,\mathrm{km\,s^{-1}}$. The diameter of the throat is $D_t = 10.3$ inches and that at the exit is $D_e = 90.7$ inches. The fuel is liquid oxygen and liquid hydrogen. The two liquids are pumped with a mass ratio of 6:1 and pumped into the burning chamber.

First I compute the density of the exhaust:

$$\begin{aligned} \rho_e &= \frac{\dot{M}}{A_e u_e} \\ &= 2.03 \times 10^{-5}\,\mathrm{g\,cm^{-3}}; \end{aligned} \tag{22}$$

here, $A_e = (\pi/4)D_e^2$ is the area of the exit aperture. For $\gamma = 5/3$, I plot the LHS and RHS of Equation 21 (Figure 2). I find, $c_e = 0.45\,\mathrm{km\,s^{-1}}$ and $c_0 = 2.8\,\mathrm{km\,s^{-1}}$. Thus the Mach number at the output is about $M_e \approx 11$.

I now explore to see if I can infer the temperature of the burning chamber (since we know c_0). The speed of sound in the burning chamber is

$$c_0 = \sqrt{\frac{\gamma k_B T}{\mu m_H}} \tag{23}$$

where μm_H is the mean molecular weight of the gas (m_H is the mass of H atom) and T is the temperature.

The fuel consists of the following mixture: 6 grams of O_2 to every gram of H_2. Following burning, we have H_2O and atomic hydrogen. By numbers we have 3/8 of single oxygen to every atom of hydrogen. Following burning we have by number: 3/8 (H_2O, $\mu = 18$) and 1/4 (H, $\mu = 1$) and thus

$$\mu = \frac{18 \times 3/8 + 1/4}{3/8 + 1/4} = 11.2. \tag{24}$$

If the hydrogen has recombined then $\mu = 14$. For $\gamma = 5/3$, $\mu = 11.2$ and given $c_0 = 2.8\,\mathrm{km\,s^{-1}}$ we obtain $T = 6550\,\mathrm{K}$.

However, the use of $\gamma = 5/3$ is questionable. At the high temperatures in the burning chamber as well as flow downstream the rotational-vibrations bands of molecular hydrogen

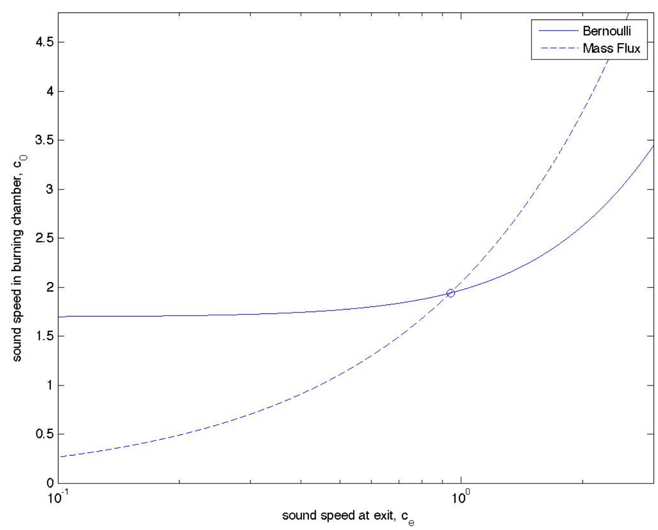

FIGURE 2. SSEM: The speed of sound at the exhaust, c_e, versus the speed of sound in the burning chamber, c_0. The curves are for the parameters relevant to the SSEM. The solid curve is the square root of the right side of Equation 21 and the dotted line is the square root of the right side of the same Equation.

[1-0 S(1), 2.1 μ, 2-1 S(1), 2.2 μm; see Black & Dalgarno] and water (see Barber et al. 2005, MNRAS) will be excited.

A perusal of the relevant NASA web pages show the following measurements: an output Mach speed of 5.5 and a temperature of 3300 K in the combustion chamber.

Adopting $\gamma = 5/4$ we find $c_0 = 1.94\,\mathrm{km\,s^{-1}}$ and $c_e = 0.94\,\mathrm{km\,s^{-1}}$ and thus $M_e = 5.1$. This compares favorably with the measured combustion temperature and measured Mach speed.

Appendix A. Rocket Nozzles

The simplest nozzle is a cone with a half-opening angle, α attached to a combustion chamber (see Figure 3). Conical nozzles yield nearly uniform exit velocity. However, there is flow divergence since the flow angle varies from 0 (on axis) to α (at the edge). Thus the the thrust is reduced.

A typical nozzle has $\alpha = 15°$. Incidentally, the length of any nozzle type is commonly referenced to the length of a 15° cone having the same nozzle area ratio. These nozzles are not used in practice (finding their greatest value for homework problems.)

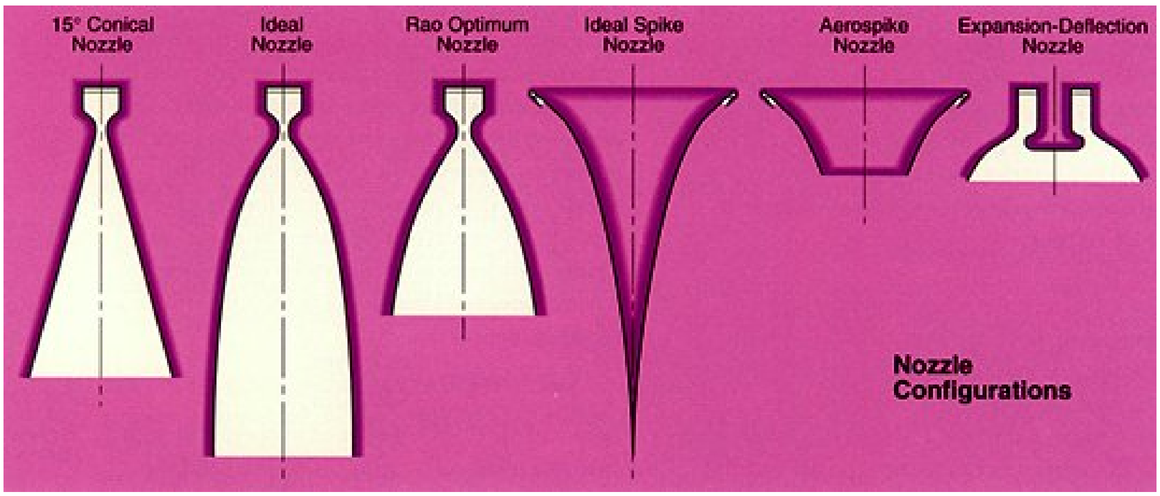

Figure 3. Types of Nozzles.

The "Rao" (after Gadicherla V. R. Rao of Rockwell Inc) nozzle uses an approximately parabolic cross-section and is of shorter length relative to a simple conical nozzle (for a given burn rate and thrust).

A cartoon view of a liquid rocket engine is shown in Figure 4. For big rockets such as the Space Shuttle, the fuel is hydrogen and the oxidizer is oxygen. These two liquids are pumped into the combustion chamber in the ratio of 1:6.[2]

The Space Shuttle Main Engine (SSME; see Figure 5) is described in [1]: "The SSME nozzle is 10.3 inches in diameter at the throat, increasing to 90.7 inches at the nozzle

[2]The primary reaction is $H_2+O_2=H_2O+O$. The density of liquid hydrogen is small, $\rho_{H_2} = 0.07\,\mathrm{g\,cm^{-3}}$; in contrast, the density of liquid oxygen is $1.14\,\mathrm{gm\,cm^{-3}}$. However, the number density of H_2 is $N_A\rho_{H_2}/\mu = 2.1\times10^{22}\,\mathrm{cm^{-3}}$ where N_A is Avogadro's number and $\mu = 2$ is the number of Daltons for molecular hydrogen. Likewise for O (single atom) the number density is $4.3\times10^{22}\,\mathrm{cm^{-3}}$. From this I deduce that the capacity of the hydrogen need only be twice that for the oxygen tank. The fuel ratio is conventionally quoted as the ratio of the oxidizer to the fuel and the units are pounds or kilograms. The ratio is expected to be 8:1 but in practice the ratio is set to 6.03:1. All the oxygen will be consumed. The unburnt hydrogen reduces the molecular weight of the exhaust which apparently reduces the turbulence that is created along the nozzle surface.

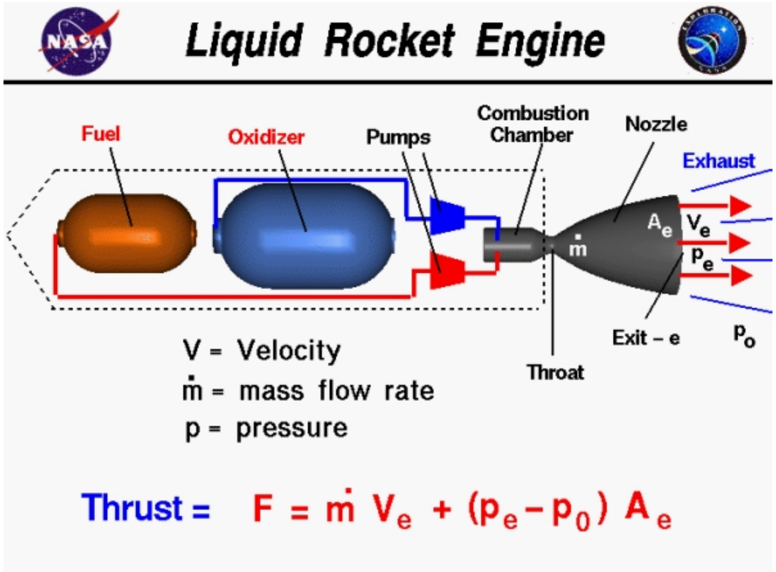

FIGURE 4. Cartoon view of a liquid engine rocket.

exit over a length of 121 inches. At 100 percent power level, propellants flow through the nozzle at a rate of 1,035 pounds per second. The nozzle accelerates the combustion products to 17,000 feet per second at the nozzle exit, generating 470,000 pounds of thrust at vacuum. Because the last one percent of SSME thrust at a fixed mass flow rate translates to about 5,000 pounds of shuttle payload, high priority was placed on nozzle design and performance."

Note one but three such engines are needed to provide the necessary lift-off thrust for the Space Shuttle. Each SSME can be throttled over a range of 65% to 109% (!) in increments of 1%.

Despite the US being a powerhouse in technology there are pockets who continue to live in the previous century.[3] To start with one pound is about 453 g. Thus a burn rate of 1035 pounds per second is $\dot{M} \approx 468\,\mathrm{kg\,s^{-1}}$. Next, one inch is exactly 2.54 cm and one foot is 12 inches. Thus one foot per second is 30.48 $\mathrm{cm\,s^{-1}}$. The exit speed is thus 5.18 $\mathrm{km\,s^{-1}}$ (cf. the speed of sound on a balmy day is about $0.343\,\mathrm{km\,s^{-1}}$).[4]

Finally, the SI unit for force is Newton ($\mathrm{kg\,m\,s^{-2}}$) and the CGS unit is dyne. In US aerospace industry force or *thrust* is quoted in "lbf". One pound force is the force due to Earth's gravity at sea level on one pound of matter. Thus $1\ \mathrm{lbf} = 0.453\,\mathrm{kg} \times 9.8\,\mathrm{m\,s^{-2}} = 4.439 \times 10^5$ dyne or 4.439 Newton. Thus the thrust generated by SSME is 2.1×10^6 N.

[3]with attendant disaster – the ill-fated Mars Climate Orbiter.

[4]Another reference quotes the exit speed (with respect to the nozzle) as 4.4 $\mathrm{km\,s^{-1}}$.

FIGURE 5. SSME Nozzle at sea-level pressure. The shape of the plume varies with the ambient pressure. Next time you see the Shuttle being launched look for the change in shape (cylindrical with Mach disk at launch; gradually the diameter of the cylinder should increase and the Mach disk should disappear; finally the plume should look a cupola.)

References:

[1] http://www.k-makris.gr/RocketTechnology/Nozzle_Design/nozzle_design.htm
[2] The article quoted in [1] is a reproduction of an article by
R.A. O'Leary & J. E. Beck for the Boeing Engineering (Spring 1992 issue).
See http://www.engineeringatboeing.com/.
[3] An Introduction to Stellar Winds by H.J.G.L.M. Lamers & J.P. Cassinelli