

# Ay 102/126 Interstellar Medium

## Problem Set 1

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### 1 Strömgen Sphere

For an O6 star ( $T_* = 3.8 \times 10^4$  K,  $R_* = 5 \times 10^{11}$  cm,  $M_* = 50 M_\odot$ ) embedded in a cloud of atomic hydrogen with density  $n_{\text{H}} = 10^2 \text{ cm}^{-3}$  and temperature  $T_0 = 25$  K, calculate

- a) Calculate the production rate of Lyman continuum photons assuming a blackbody spectrum.

Lyman continuum photons are those with energy greater than 13.6 eV. We can calculate the total rate of production of these photons by numerical integration. The number of photons emitted in over all solid angles per unit time per unit area at the surface of the star is given by,

$$dq_\nu = \pi \frac{B_\nu(T_*)}{h\nu} \quad (1)$$

The total number of photons emitted by the star per unit time is then given by,

$$Q_* = 4\pi R_*^2 \times \int_{\nu_0}^{\infty} \frac{\pi B_\nu(T_*)}{h\nu} d\nu = 4.75 \times 10^{48} \text{ s}^{-1} \quad (2)$$

For those who prefer to have their own codes instead of numerical packages, a perl code for calculating this is linked from the course website.

We can also derive an analytic expression for this - see for eg Rybicki and Lightman Chapter 1. In our range of interest, we have:

$$\frac{h\nu}{kT_*} \geq \frac{h\nu_0}{kT_*} = \frac{13.6 \text{ eV}}{8.62 \times 10^{-5} \text{ eV K}^{-1} \cdot 3.8 \times 10^4 \text{ K}} = 4.2 \quad (3)$$

Hence we can use Weins approximation to the blackbody curve,

$$B_\nu \simeq \frac{2h\nu^3}{c^2} \exp\left(\frac{-h\nu}{kT}\right) \quad (4)$$

This can be integrated analytically by parts to get,

$$\begin{aligned} Q_* &= 4\pi R_*^2 \times \int_{\nu_0}^{\infty} \frac{\pi B_\nu(T_*)}{h\nu} d\nu \\ Q_* &= \frac{8\pi^2 R_*^2}{c^2} \left(\frac{kT_*}{h}\right)^3 \int_{\left(\frac{h\nu_0}{kT_*}\right)}^{\infty} \left(\frac{h\nu}{kT_*}\right)^2 \exp\left(\frac{h\nu}{kT_*}\right) d\left(\frac{h\nu}{kT_*}\right) \\ Q_* &= \frac{8\pi^2 R_*^2}{c^2} \left(\frac{kT_*}{h}\right)^3 [-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}] \Big|_{\left(\frac{h\nu_0}{kT_*}\right)}^{\infty} \end{aligned} \quad (5)$$

This approximation gives the answer,  $Q_* = 4.71 \times 10^{48}$ , very close to our numerical solution.

**b) Calculate the size of the Strömgen sphere (when it is first ionized).**

The Strömgen sphere is ionization bounded, so we can assume it is optically thick to Lyman continuum photons. Hence, all  $Q_*$  Lyman continuum photons can be assumed to be absorbed at  $R < R_s$ , with no photons reaching  $R > R_s$ . These photons balance the reionize any neutral atoms formed due to recombination within the Strömgen sphere. Any recombination to the  $n = 1$  level will add to the diffuse radiation field and ionize another atom. So must use  $\alpha_B$  while calculating the effective rate of recombination. Thus we have,

$$Q_* = \frac{4\pi}{3} R_*^3 \times n_e n_p \alpha_B \quad (6)$$

Using  $\alpha_B = 2.6 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1}$ , we get:

$$R_s = 7.6 \times 10^{18} \text{ cm} = 2.5 \text{ pc} \quad (7)$$

Note that  $\alpha_B$  is approximated as (*Osterbrock, Table 2.1*),

$$\alpha_B = 2.59 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1} \times \left( \frac{T}{10^4 \text{ K}} \right)^{-0.5} \quad (8)$$

The typical temperature in such a nebula is  $\sim 10^4 \text{ K}$ , which justifies the value of  $\alpha_B$  we have used. Let us calculate the mean temperature of this Strömgen region. The excess energy from the Lyman continuum photons is transferred to the electrons as kinetic energy. The electrons then thermalize with the protons to equipartition the energy. Effectively, the mean excess energy per photon is transferred to 2 particles.

For the Strömgen sphere we are considering, the excess energy  $E$  is:

$$E = 4\pi R_*^2 \int_{\nu_0}^{\infty} \frac{\pi B_\nu(T_*)}{h\nu} (h\nu - 13.6 \text{ eV}) d\nu = 3.6 \times 10^{37} \text{ erg s}^{-1} \quad (9)$$

Thus the mean energy per photon is,

$$\bar{E} = \frac{E}{Q_*} = 7.6 \times 10^{-12} \text{ erg} = 4.8 \text{ eV} \quad (10)$$

Finally, we calculate the effective temperature, using:

$$2 \times \frac{3}{2} k T_{eff} = \bar{E} \quad (11)$$

$$T_{eff} = 1.8 \times 10^4 \text{ K} \quad (12)$$

This is not too far from our assumed value of  $10^4 \text{ K}$ , hence using  $\alpha_B = 2.6 \times 10^{-13}$  does not affect the answer drastically. Note that in calculating this temperature, we are neglecting other effects. For example, energy is transferred by collisions to metal ions, which radiate away the energy at longer wavelengths. Energy loss by Bremsstrahlung is significant only at temperatures  $\gtrsim 10^6 \text{ K}$  and may be neglected here.

- c) **The ionization fraction is defined as  $y = n_p/n_H$ . Equate the rates of ionization and recombination to calculate the ionization fraction at half this Strömgen radius.**

Let us make an on-the-spot approximation, that the rates of ionization and recombination per unit volume are equal at each point in the Strömgen sphere. For simplicity, let us ignore the “hardening” of radiation as a function of radius. Another factor to be considered is the reduction in flux of ionizing photons due to absorption of photons by intervening material. Let  $p$  be the fraction of photons that reach a distance  $R$  from the star. Then we can equate the rate of ionization and rate of recombination per unit volume, to get:

$$\int_{\nu_0}^{\infty} \left( \frac{\pi B_{\nu}}{h\nu} \times p \times \frac{4\pi R_*^2}{4\pi R^2} \right) n_H a_{\nu} d\nu = n_e n_p \alpha_B \quad (13)$$

$$p = 1 - \left( \frac{R}{R_s} \right)^3 = 0.875 \quad (14)$$

Since we are ignoring hardening and assuming  $a_{\nu}$  to be constant (say  $a$ ), we can use  $Q_*$  from Equation 2 to get:

$$p(1-y)n_H a \frac{Q_*}{4\pi R^2} = (yn_H)^2 \alpha_B \quad (15)$$

Solving for  $y$ , we get

$$\frac{1-y}{y^2} = 1.8 \times 10^{-4}$$

The only positive solution to this quadratic is  $y = 0.9998$ . Hence ionization is essentially complete even at  $0.5R_s$ . Figure 1 shows ionization fraction as a function of radius within the Strömgen sphere. The ionization is almost complete upto the edge of the sphere where there is a sudden transition from ionized to neutral medium.

- d) **The ionization timescale is the inverse of the rate of ionization. Similarly the recombination timescale is the inverse of the rate of recombination. Evaluate the timescales for ionization of a neutral hydrogen atom and for recombination of a hydrogen ion at  $0.1R_s$  and  $0.9R_s$ .**

As defined above, the recombination timescale  $\tau_r$  is given by,

$$\tau_r = \frac{n_p}{n_p n_e \alpha_B} = \frac{1}{yn_H \alpha_B} \quad (16)$$

The ionization fraction can be calculated as in part c) for any desired radius. At  $0.1R_s$  we have  $y \simeq 1$  and at  $0.9R_s$  we get  $y = 0.997$ . We use  $n_e = n_p = yn_H$ ,  $\alpha_B = 2.6 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1}$  and  $n_H = 100 \text{ cm}^{-3}$  to get,

$$\tau_r(r = 0.1R_s) = 3.8 \times 10^{10} \text{ s} \simeq 1200 \text{ yrs} \quad (17)$$

$$\tau_r(r = 0.9R_s) = 3.9 \times 10^{10} \text{ s} \simeq 1200 \text{ yrs} \quad (18)$$

The difference in these timescales is very small as the gas is almost fully ionized in the entire Strömgen sphere.

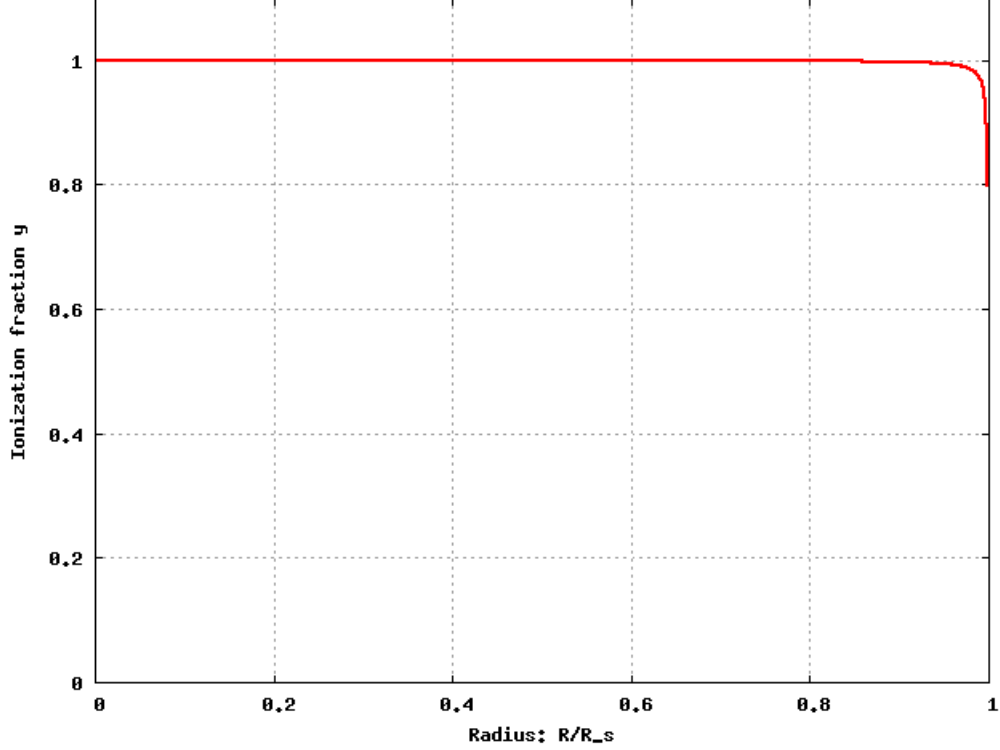


Figure 1: Ionization fraction ( $y$ ) of Hydrogen in a Strömgen sphere as a function of radius. The entire Strömgen is almost fully ionized.

The ionization timescale is the inverse of the rate of ionization. Thus we have,

$$\tau_i = \frac{1}{R_i} = \frac{1}{\langle a_\nu \rangle N_\nu} = \frac{4\pi r^2}{\langle a_\nu \rangle p Q_*} \quad (19)$$

Here  $p$  is the fraction of ionizing photons that reach the radius  $r$ . Plugging in all the numbers, we get

$$\tau_i(r = 0.1R_S) = 2.4 \times 10^5 \text{ s} = 2.8 \text{ days} \quad (20)$$

$$\tau_i(r = 0.9R_S) = 7.2 \times 10^7 \text{ s} = 2.3 \text{ yrs} \quad (21)$$

Another method to calculate the rate of ionization is to note that the *volumetric* rates of ionization and recombination are equal. Hence we have,

$$\frac{n_0}{\tau_i} = \frac{n_+}{\tau_r} \quad (22)$$

Using this gives results which are in good agreement with the calculations above.

**e) Estimate the thickness of the ionization front (the transition thickness from mostly HII to mostly HI).**

The thickness of the ionizing front is the mean free path of an ionizing photon in this medium. Let us define the edge of the Strömgen sphere as the place where half the gas is ionized. Then we have,

$$\Delta r = \frac{1}{0.5 n_H a_\nu} = 3.2 \times 10^{15} \text{ cm} = 0.001 \text{ pc} \quad (23)$$

Note that we have used  $a_\nu = 6.3 \times 10^{-18} \text{ cm}^2$ , but in reality the effective  $a_\nu$  decreases as a function of radius due to the hardening of radiation. Thus, our estimate of the thickness is actually a lower bound.

**f) Calculate the eventual size of the Strömgen sphere when it has reached pressure equilibrium with the ambient gas.**

Let us use the subscripts 0 and  $f$  to denote the initial and final conditions in the Strömgen sphere respectively. For balancing pressure, we have:

$$(n_p + n_e)_f k T_f = n_0 k T_0 \quad (24)$$

We know that  $T_0 = 25 \text{ K}$ , and assume a generic  $T_f = 10^4 \text{ K}$ . Further, we know  $n_p = n_e$ . This gives,

$$n_f = \frac{n_0 T_0}{2 T_f} = 0.125 \text{ cm}^{-3} \quad (25)$$

Solving again for the radius of the Strömgen sphere, we get:

$$R_S = \left( \frac{3Q_*}{4\pi n_f^2 \alpha_B} \right)^{1/3} = R_{S,0} \left( \frac{n_i}{n_f} \right)^{2/3} \quad (26)$$

Hence the final size of the Strömgen sphere is  $R_S = 6.5 \times 10^{20} \text{ cm} = 211 \text{ pc}$ .

## 2 Strömgen length

**The intergalactic background at a redshift of  $z = 3$  has an intensity at  $h\nu_0 = 13.6 \text{ eV}$  of  $J_\nu(\nu_0) = 10^{-21} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1} \text{ ster}^{-1}$ , produced by the light of distant quasars and young galaxies.**

**a) Estimate the Strömgen length of a young galaxy's outer disk, where  $n_{\text{H}} \simeq 0.1 \text{ cm}^{-3}$ .**

The spectrum of the ionizing radiation is characterized by,

$$J_\nu(\nu_0) = 10^{-21} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1} \text{ ster}^{-1}$$

$$J_\nu = J_\nu(\nu_0) \left( \frac{\nu}{\nu_0} \right)^{-1.5}$$

The total flux of ionizing photons can be calculated by integrating this over all ionizing frequencies:

$$Q = \int_{\nu_0}^{\infty} \frac{\pi J_\nu d\nu}{h\nu} = \frac{\pi}{\nu_0^{-1.5}} \times \frac{J_\nu}{h} \int_{\nu_0}^{\infty} \nu^{-2.5} d\nu \quad (27)$$

The factor of  $\pi$  in Equation 27 comes from integrating over the solid angle in which radiation is incident on the disc. The lowest energy ionizing photons has energy 13.6 eV, so  $\nu_0 = 3.3 \times 10^{15} \text{ Hz}$ . This is a number worth remembering ! Substituting the values, we get:

$$Q = 3.2 \times 10^5 \text{ photons cm}^{-2} \text{ s}^{-1} \quad (28)$$

In equilibrium, the flux of ionizing photons equals the rate of recombination. Thus we have,

$$Q = n^2 \alpha_B l_S \quad (29)$$

where  $l_S$  is the Strömgen length. Substituting the numbers,

$$l_S = \frac{Q}{n^2 \alpha_B} = 1.2 \times 10^{20} \text{ cm} = 40 \text{ pc} \quad (30)$$

**b) What is the maximum total column density of hydrogen which can be kept ionized by the background light?**

The total column density which can be kept ionized is simply given by,

$$l_S \cdot n_H = 1.2 \times 10^{19} \text{ atoms cm}^{-2} \quad (31)$$

### 3 Forming Stars

Stars are believed to form by gravitational collapse of dense interstellar clouds. Suppose that the infall of matter continues after the O6 star in problem 1 has turned on.

**a) What is the accretion rate ( $M_\odot, \text{yr}^{-1}$ ) required for the HII region to be entirely suppressed?**

Since we are assuming a pure Hydrogen cloud, the accretion rate must just equal the rate of production of Lyman photons at the lowest accretion rate where the Strömgen region is fully suppressed. This gives,

$$\dot{n} = Q_*$$

$$\dot{M} = \frac{\dot{n} m_H}{M_\odot} = 0.13 M_\odot \text{ yr}^{-1} \quad (32)$$

**b) What accretion rate will limit the size of the HII region to 0.15 pc? You may assume that the accreting gas follows a free-fall velocity law.**

With the Strömgen sphere limited to 0.15 pc, the Lyman photons are being used up in two ways - to balance the recombinations happening within 0.15 pc, and to ionize material that is falling in through the boundary of the Strömgen sphere.

For proceeding with calculations, we first need to find density as a function of radius. At a radius  $r$  we have,

$$\dot{M} = 4\pi r^2 \rho(r) v(r) \quad (33)$$

where  $v(r)$  is the velocity of infalling gas, which is given by the free-fall law:

$$v(r) = \sqrt{\frac{2GM_*}{r}} \quad (34)$$

So the density profile is given by,

$$\rho(r) = \frac{\dot{M}}{4\pi r^{3/2} \sqrt{2GM_*}} \quad (35)$$

Now let us proceed to calculate the net recombination rate within the Strömgen sphere. Let us denote the rate by  $K$  (for lack of a better letter).

$$K = \int_{R_*}^{R_S} n^2 \alpha_B \cdot 4\pi r^2 dr \quad (36)$$

$$K = \frac{4\pi\alpha_B}{m_p^2} \int_{R_*}^{R_S} \rho^2 r^2 dr = \frac{\alpha_B \dot{M}^2}{8\pi G M_* m_p^2} \int_{R_*}^{R_S} \frac{r^2}{r^3} dr$$

where the second step results from using Equation 35. Substituting the numbers (using CGS units), we get:

$$K = 7.75 \times 10^6 \dot{M}^2 \quad (37)$$

The remaining photons are used up to ionize material that is falling in through the boundary of the Strömgen sphere at 0.15 pc. Hence, we get:

$$Q = K + \frac{\dot{M}}{m_p} \quad (38)$$

Again substituting values in CGS units, we get,

$$4.8 \times 10^{48} = 7.8 \times 10^6 \dot{M}^2 + 6 \times 10^{23} \dot{M} \quad (39)$$

We see that the first term by itself will give  $\dot{M} \simeq 10^{21}$ , for which the contribution from the second term is negligible. The second term by itself needs  $\dot{M} \simeq 10^{25}$ , which will cause the first term to overshoot. Or of course, you can solve the full quadratic. The final answer is,

$$\dot{M} = 7.8 \times 10^{20} \text{ g s}^{-1} = 1.2 \times 10^{-5} M_\odot \text{ yr}^{-1} \quad (40)$$

## 4 Stellar Wind

**A stellar wind with density distribution  $n(r) = n_0(r/r_0)^{-2}$  is being ionized by a central star with a total Lyman continuum photon output of  $Q$ .**

**a) Calculate the number of ionizing photons remaining as a function of radius.**

The ionizing photons emitted by the star are used up in ionizing the stellar wind. In steady state, at any given place, the rate of ionization equals the rate of recombination. Hence we can integrate outwards from the stellar surface and write,

$$Q(R) = Q_0 - \int_{R_*}^R n(r)^2 \alpha_B \cdot 4\pi r^2 dr \quad (41)$$

We know that the density profile is,

$$n(r) = n_0 \left( \frac{r}{r_0} \right)^{-2} \quad (42)$$

where  $r_0 = R_*$ . Substituting this and integrating, we get:

$$Q(R) = Q_0 - 4\pi n_0^2 r_0^4 \alpha_B \left( \frac{1}{r_0} - \frac{1}{R} \right) \quad (43)$$

b) Find the minimum value of  $Q$  for which the entire stellar wind is ionized.

The minimum value of  $Q$  for ionizing the entire wind will be the one where  $Q(R) \rightarrow 0$  as  $R \rightarrow \infty$ . Plugging this condition into Equation 43, we get:

$$Q_{min} = 4\pi n_0^2 r_0^3 \alpha_B \quad (44)$$

c) For  $Q$  smaller than this critical value, show that the Strömgen radius is given by,  $r_s = \frac{r_0}{1-Q/Q_1}$  where  $Q_1 = 4\pi r_0^3 n_0^2 \alpha_B$ .

For  $Q < Q_{min}$ , the boundary of the Strömgen sphere is where all the ionizing photons emitted by the star are used up. Thus, we put the condition that  $Q(R_S) = 0$  in Equation 43, to get:

$$Q_0 = 4\pi n_0^2 r_0^4 \alpha_B \left( \frac{1}{r_0} - \frac{1}{R_S} \right) \quad (45)$$

After some quick algebra, we arrive at:

$$R_S = \frac{r_0}{1 - Q/Q_1} \quad (46)$$

where  $Q_1 = Q_{min} = 4\pi n_0^2 r_0^3 \alpha_B$ .

d) For a star of temperature 50000 K and a luminosity of 50000  $L_\odot$ , calculate the maximum mass loss rate of the stellar wind that can be fully ionized by the star. Assume the stellar wind begins at the stellar surface, has a gas temperature of 10000 K, and is expanding at a speed of 2000  $km\ s^{-1}$ . *Kwok, problem 4.3*

There is a small catch in this problem that we shouldnt overlook. The stellar wind velocity need not be the same as escape velocity – in this case, the velocity is actually *greater* than the escape velocity. Let  $v_*$  denote the velocity of the wind at the stellar surface. The wind velocity at a distance  $r$  from the center of the star can then be calculated by conserving energy:

$$\frac{1}{2}v_*^2 - \frac{GM_*}{r_*} = \frac{1}{2}v(r)^2 - \frac{GM_*}{r}$$

We are assuming that the energy input due to ionizing photons etc, and energy losses by radiation, change only the random thermal component of the motion of gas particles, and do not affect the directed velocity. Thus we get,

$$v(r) = \sqrt{2GM_* \left( \frac{1}{r} - \frac{1}{r_*} \right) + v_{r_*}^2} \quad (47)$$

Now, we note that the wind flux is constant, hence  $4\pi r^2 \cdot v(r) \cdot n$  is a constant. This gives,

$$n(r) = \frac{n_* v_* r_*^2}{r^2 v(r)} \quad (48)$$

Now to calculate for the maximum mass loss rate of the stellar wind that can be fully ionized, we have to solve an integral similar to 41:

$$Q = \int_{r_*}^{\infty} n(r)^2 \alpha_B \cdot 4\pi r^2 dr \quad (49)$$

$$Q = 4\pi \alpha_B \int_{r_*}^{\infty} \frac{n_*^2 v_*^2 r_*^4 r^2}{r^4 \left( 2GM_* \left( \frac{1}{r} - \frac{1}{r_*} \right) + v_{r_*}^2 \right)} dr \quad (50)$$

Complicated as it may look, this expression does have an analytic solution ! For those interested, we've added it as an appendix. For now, lets be lazy and assume we get an answer from your favourite numerical package. That yeilds,

$$n_* = 1.5 \times 10^{14} \text{ cm}^{-3} \quad (51)$$

The maximum mass loss rate is then given by,

$$\dot{M}_{max} = 4\pi r_*^2 v_* n_* m_H = 2.9 \times 10^{22} \text{ g s}^{-1} = 4.5 \times 10^{-4} M_{\odot} \text{ yr}^{-1} \quad (52)$$

## Appendix: Solving Equation 50

Equation 50 is the integral of a reciprocal of a quadratic without a constant term. We can use the following result to integrate it:

$$\int \frac{dx}{ax^2 + bx} = \frac{1}{b} \ln \left| \frac{ax}{ax + b} \right| \quad (53)$$

We can recast Equation 50 as,

$$Q = \frac{4\pi \alpha_B n_*^2 v_*^2 r_*^4}{2GM_*} r_* \int_{\infty}^{r_*} \frac{dr}{\left( 1 - \frac{v_*^2 r_*}{2GM_*} \right) \cdot r^2 + (-r_*) \cdot r} \quad (54)$$

Thus in Equation 53 we can put  $x = r$ ,  $a = 1 - \frac{v_*^2 r_*}{2GM_*}$ , and  $b = -r_*$ . With these values, we see that the ln term vanishes as  $r \rightarrow \infty$ . Thus we can substitute values in Equation 53 to get,

$$Q = \left[ \frac{4\pi \alpha_B n_*^2 v_*^2 r_*^4}{2GM_*} \ln \left| \frac{\left( 1 - \frac{v_*^2 r_*}{2GM_*} \right) r_*}{\left( 1 - \frac{v_*^2 r_*}{2GM_*} \right) r_* - r_*} \right| \right]_{r=r_*} \quad (55)$$

Now we note that  $a = 1 - \frac{v_*^2 r_*}{2GM_*} \leq 0$  if the wind escapes from the star. (Verify this by checking the escape velocity at the stellar surface). Let us evaluate the ln term now:

$$\ln | \text{blah} | = \ln \left| \frac{2GM_*}{v_*^2 r_*} - 1 \right| = 7.36 \quad (56)$$

Ah ! We could as well have applied a very useful “order of magnitude”<sup>1</sup> trick here: log of any big number is 10, to within an order of magnitude. Finally, plugging in the other numbers, we arrive at the answer in Equation 51.

For those who actually read the solution this far: dont worry, nobody thought of this aspect of the problem, and just assumed that the wind was streaming out at escape velocity. Which of course means there was no question of solving this integral: neither by software nor analytically !!!

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<sup>1</sup>We learnt it from Sterl Phinney