Ay 102/126 Interstellar Medium
Problem Set 2

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1 Neon lines

(Modified from Kwok Ch. 5, problem 7) The 12.8 μm line is a strong emission line in the infrared spectrum of emission nebulae. This line is identified as the $^2\mathbf{P_{1/2}}$ (upper) to $^2\mathbf{P_{3/2}}$ (lower) transition of singly ionized Ne. Given the collisional strength $\mathbf{\Omega = 0.37}$, and the spontaneous decay rate $\mathbf{A_{ul} = 8.6 \times 10^{-3}\,s^{-1}}$, and assuming that this line is collisionally excited calculate:

a) The energy of this transition.

The energy of the transition is simply given by:

$$E = \frac{hc}{\lambda} = 1.55 \times 10^{-13}\,\text{ergs} = 0.097\,\text{eV} \tag{1}$$

b) The upward collision rate constant $\mathbf{C_{lu}}$ and downward collision rate constant $\mathbf{C_{ul}}$ if the kinetic temperature of the gas is $\mathbf{T = 10^4}\,\text{K}$.

The upward collision rate constant C_{lu} is given by,

$$\begin{aligned} C_{lu} &= \frac{8.629 \times 10^{-6}\,\text{cm}^3\,\text{s}^{-1}}{\sqrt{T/\text{K}}} \cdot \frac{\Omega}{g_l} e^{-E_{ul}/kT_e} \\ &= 7.13 \times 10^{-9}\,\text{cm}^3\,\text{s}^{-1} \end{aligned} \tag{2}$$

where g_l is the degenracy of the $^2P_{3/2}$ state, given by $g_l = 2J + 1 = 4$. $\Omega = 0.37$. The downward collision rate constant is given by,

$$\begin{aligned} C_{ul} &= C_{lu} \frac{g_l}{g_u} e^{E_{ul}/kT_e} \\ &= 1.60 \times 10^{-8}\,\text{cm}^3\,\text{s}^{-1} \end{aligned} \tag{3}$$

here, $g_u = 2$ for the $^2P_{1/2}$ state.

c) The critical density for this transition.

The critical density is defined as the density at which the rate of collisional de–excitation equals the rate of spontaneous de–excitation. The critical density for this Neon transition is given by,

$$n_c = \frac{A_{ul}}{C_{ul}} = 5.39 \times 10^5\,\text{cm}^{-3} \tag{4}$$

d) Calculate the ratio of the level populations n_u/n_l for $n_e = 10^4\,\mathrm{cm^{-3}}$.

The level populations will attain a steady state when:

$$n_u \cdot (A_{ul} + C_{ul} n_e) = n_l \cdot C_{lu} n_e$$

Thus, the ratio of the level populations is given by:

$$\frac{n_u}{n_l} = \frac{C_{lu} n_e}{A_{ul} + C_{ul} n_e} = 8.1 \times 10^{-3} \tag{5}$$

e) For a $10^4\,\mathrm{M_\odot}$ cloud of HII at 1 kpc with density $n_H = 1\,\mathrm{cm^{-3}}$, what is the line flux at the earth (in $\mathrm{erg\,cm^{-2}\,s^{-1}}$)? Assume the same temperature as above. You may ignore optical depth effects. Assume $N_{Ne}/N_H = 10^{-4}$, taken from typical values in Dopita table 10.2.

Note: Problem corrected as per emails, we are considering a HII region and not a HI region.

To calculate the Flux at earth, we have to calculate the total luminosity of the cloud. Let us assume that the ionization fraction of Neon is neglibible, and all the electrons come from HII. The density of Neon is $n_{\mathrm{Ne}} = 10^{-4}\,\mathrm{cm^{-3}}$ which is much lower than the critical density calculated in Part c). Thus, we can ignore collisional de–excitation, and assume all de–excitations are radiative. The total number of radiative de–excitations per second is given by,

$$\begin{aligned} N &= (\text{Number of excited Neon atoms}) \times A_{ul} \\ &= \frac{n_u}{n_{\mathrm{Ne}}} \cdot \frac{n_{\mathrm{Ne}}}{n_H} \cdot \frac{M_{cloud}}{\mu m_H} \cdot A_{ul} \end{aligned} \tag{6}$$

where μ is the mean atomic mass of the HII region. We further assume that $n_u + n_l = n_{\mathrm{Ne}}$. Then from Equation 5 we can write,

$$\frac{n_u}{n_{\mathrm{Ne}}} = \frac{C_{lu} n_e}{C_{lu} n_e + A_{ul} + C_{ul} n_e} \tag{7}$$

The electrons come from Hydrogen, so $n_e = 1\,\mathrm{cm^{-3}}$. Substituting values of C_{lu} and C_{ul} from Equations 2 and 3 into Equation 7 we get:

$$\frac{n_u}{n_{\mathrm{Ne}}} = 8.29 \times 10^{-7} \tag{8}$$

Now we substitute this value in Equation 6. Note that $\mu \simeq 0.7$ for an HII region. This gives the number of emissions in the cloud per second to be,

$$N = 1.22 \times 10^{49}\,\mathrm{s^{-1}} \tag{9}$$

To obtain the flux at earth, we now multiply by the energy of the photon (Equation 1), and account for the distance between the cloud and earth, to get:

$$\begin{aligned} F &= E \cdot \frac{N}{4\pi d^2} \\ &= 1.58 \times 10^{-8}\,\mathrm{erg\,cm^{-2}\,s^{-1}} \end{aligned} \tag{10}$$

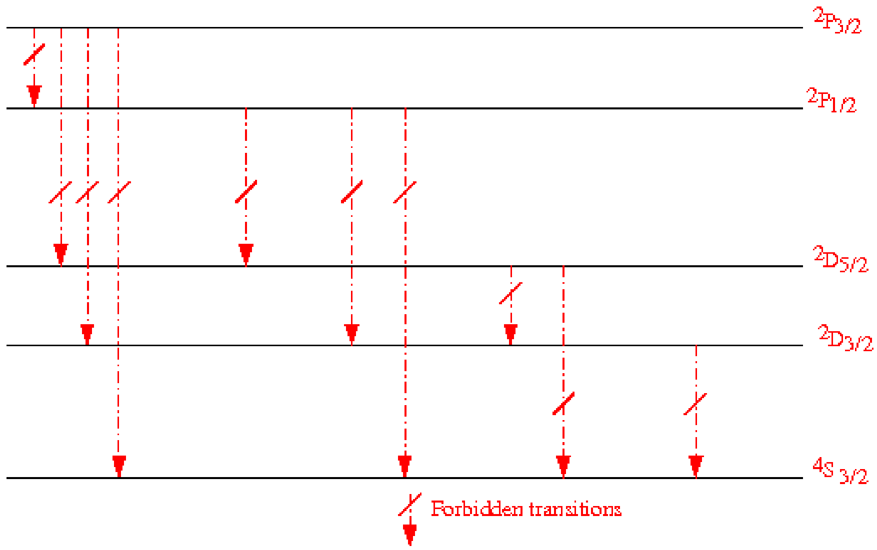

Figure 1: Energy Level Diagram for SII atom

2 SII lines

Consider the SII levels given in Osterbrock's Table 3.13

a) Draw an energy level diagram for the P, D, and S levels and show all permitted radiative transitions based on this table.

Forbidden and permitted transitions are determined according to the selection rules *(See, for example, Kwok, p. 104)*. The Figure 1 shows the transitions for the S, P, D levels of the SII atom. The *forbidden* transitions are forbidden electric dipole transitions. However, atoms do radiate energy at these wavelengths by undergoing magnetic dipole and/or electric quadrupole transitions. These transitions are less probable, and the lines are weaker than the electric dipole lines.

b) For the ^{2}D and ^{4}S levels, calculate the relative populations at $\mathbf{n_e = 10, 10^3}$, and $\mathbf{10^5\,cm^{-3}}$. Do not include the ^{2}P levels, and assume $\mathbf{T_e = 10^4\,K}$. The values of $\mathbf{\Omega_{ji}}$ required in Kwok are given in Osterbrock, Table 3.7.

We are considering a 3 level system. Let us denote the $^2D_{5/2}$, $^2D_{3/2}$ and $^4S_{3/2}$ levels as level 3, 2 and 1 respectively. Let us first look up the required coefficients from Osterborck (see Table 1)
The collisional transition rates are given by,

$$q_{ul} = \frac{8.629 \times 10^{-6}}{T^{1/2}} \cdot \frac{\gamma(l,u)}{w_u} \tag{11}$$

$$q_{lu} = \frac{w_u}{w_l} q_{ul} e^{-\Delta E/kT} \tag{12}$$

Table 1: Transition probabilities: A (s^{-1})

$^2D_{5/2} \rightarrow ^2D_{3/2}$	$A_{32} = 3.3 \times 10^{-7}\,\mathrm{s}^{-1}$
$^2D_{5/2} \rightarrow ^4S_{3/2}$	$A_{31} = 2.6 \times 10^{-4}\,\mathrm{s}^{-1}$
$^2D_{3/2} \rightarrow ^4S_{3/2}$	$A_{21} = 8.8 \times 10^{-4}\,\mathrm{s}^{-1}$

Table 2: Collision strengths γ for SII

$^4S^0 \rightarrow^2 D^0$	$\gamma = 6.90$
$^4S_{3/2} \rightarrow^2 D_{3/2}$	$\gamma = 2.76$
$^4S_{3/2} \rightarrow^2 D_{5/2}$	$\gamma = 4.14$
$^2D_{3/2} \rightarrow^2 D_{5/2}$	$\gamma = 7.47$

Here, w is the degeneracy of the level, given by:

$$w = 2J + 1 \tag{13}$$

Again from Osterbrock, the collision strengths γ for SII are given in Table 2.
The degeneracy of the levels is calculated by Equation 13, and we have $w_1(^4S_{3/2}) = 4$, $w_2(^2D_{3/2}) = 4$ and $w_3(^2D_{5/2}) = 6$. Thus, we get the collisional rates:

$$\begin{aligned} q_{32} &= \frac{8.629 \times 10^{-6}}{\sqrt{10^4}} \cdot \frac{7.47}{6} = 1.07 \times 10^{-7}\,\text{cm}^{-3}\,\text{s}^{-1} \\ q_{21} &= \frac{8.629 \times 10^{-6}}{\sqrt{10^4}} \cdot \frac{2.76}{4} = 5.95 \times 10^{-8}\,\text{cm}^{-3}\,\text{s}^{-1} \\ q_{31} &= \frac{8.629 \times 10^{-6}}{\sqrt{10^4}} \cdot \frac{4.14}{6} = 5.95 \times 10^{-8}\,\text{cm}^{-3}\,\text{s}^{-1} \end{aligned} \tag{14}$$

The energy differences in these levels are can be calculated from the wavelengths of the transitions. The results are give in Table 3.
Using these, we can calculate the remaining collisional rates using Equation 12:

$$\begin{gathered} q_{23} = \frac{6}{4} \cdot 1.07 \times 10^{-7} \cdot e^{-6.32\times10^{-15}/kT} = 1.60 \times 10^{-7}\,\text{cm}^3\,\text{s}^{-1} \\ q_{12} = 7.00 \times 10^{-9}\,\text{cm}^3\,\text{s}^{-1} \\ q_{13} = 1.04 \times 10^{-8}\,\text{cm}^3\,\text{s}^{-1} \end{gathered} \tag{15}$$

Now that we have all the coefficients, we can write a system of linear equations to solve for the detailed balance at each level. For the first level, we have:

$$n_e n_1 q_{12} + n_e n_1 q_{13} = n_e n_2 q21 + n_e n_3 q_{31} + n_2 A_{21} + n_3 A_{31} \tag{16}$$

For the second level,

$$n_e N_2 q_{23} + n_e n_2 q_{21} + n_2 A_{21} = n_e n_1 q_{12} + n_e n_3 q_{32} + n_3 A_{32} \tag{17}$$

Now we can either write the equation for the third state, or just conserve the total number of particles. These 4 equations are not linearly independent, so using any 3 is fine. Let us use the conservation of total number of particles:

$$n_{SII} = n_1 + n_2 + n_3 \tag{18}$$

Table 3: Energy differences in SII levels

Wavelength	ΔE
$\lambda_{32} = 314.5\mu\text{m}$	$E_{32} = 6.32 \times 10^{-15}\,\text{erg}$
$\lambda_{31} = 6716.5\text{Å}$	$E_{31} = 2.96 \times 10^{-12}\,\text{erg}$
$\lambda_{21} = 6730.8\text{Å}$	$E_{21} = 2.95 \times 10^{-12}\,\text{erg}$

Table 4: Relative populations of SII levels

n_e	n_1/n_{SII}	n_2/n_{SII}	n_3/n_{SII}
$10\,\mathrm{cm}^{-3}$	0.999	0.00008	0.0004
$10^3\,\mathrm{cm}^{-3}$	0.964	0.0088	0.027
$10^5\,\mathrm{cm}^{-3}$	0.787	0.084	0.129

We can solve this by inverting the matrices manually, or by writing our own code, or by using our favourite mathematical package.... and the answer is given in Table 4.

c) Calculate the total cooling rate per SII ion at each of the above densities, including only the $^2\mathrm{D} \rightarrow {}^4\mathrm{S}$ transitions.

The luminosity due to cooling is given by,

$$L_c = \sum_i n_i \sum_{j<i} A_{ij} h\nu_{ij} \,\mathrm{erg\,cm^{-3}\,s^{-1}} \tag{19}$$

For the system we are considering (only $^2D \rightarrow {}^4S$ transitions), this gives:

$$L_c = n_2 A_{21} h\nu_{21} + n_3 A_{31} h\nu_{31} \tag{20}$$

We substitute the values of A_{ij} from Table 1 and the energy differences from Table 3 for the three values of density to get,

$$\begin{aligned} L_c(n_e = 10\,\mathrm{cm}^{-3}) &= 5.15 \times 10^{-19}\,\mathrm{erg\,s^{-1}} \\ L_c(n_e = 10^3\,\mathrm{cm}^{-3}) &= 4.35 \times 10^{-17}\,\mathrm{erg\,s^{-1}} \\ L_c(n_e = 10^5\,\mathrm{cm}^{-3}) &= 3.18 \times 10^{-16}\,\mathrm{erg\,s^{-1}} \end{aligned} \tag{21}$$

d) Calculate the excitation temperatures for the $^2\mathrm{D}_{3/2} \rightarrow {}^4\mathrm{S}_{3/2}$ transition in each of the above cases.

The $^2D_{3/2} \rightarrow {}^4S_{3/2}$ transition is from level 2 to 1. For calculating the excitation temperature, we use:

$$\begin{aligned} \frac{n_2}{n_1} &= \frac{g_2}{g_1} e^{-E_{21}/kT_{ex}} \\ T_{ex} &= \frac{-E_{21}/k}{ln\left(\frac{n_2 g_1}{n_1 g_2}\right)} \end{aligned} \tag{22}$$

Substituting the values of n_e we get,

$$\begin{aligned} T_{ex}(n_e = 10\,\mathrm{cm}^{-3}) &= 2.3 \times 10^3\,\mathrm{K} \\ T_{ex}(n_e = 10^3\,\mathrm{cm}^{-3}) &= 4.5 \times 10^3\,\mathrm{K} \\ T_{ex}(n_e = 10^5\,\mathrm{cm}^{-3}) &= 9.6 \times 10^3\,\mathrm{K} \end{aligned} \tag{23}$$

e) Discuss in a couple of sentences the asymptotic behavior of the excitation temperature and the cooling rate.

Low $\mathbf{n_e}$: At low n_e, there are too few electrons to collide with SII, and collisional excitation rates are very low. Hence, $T_x \to 0$ when $n_e \to 0$. Since the principle way of exciting SII is collisions, low n_e also implies that there are only a few atoms in the excited state. Hence, the number of radiative de–excitations is also very low. Hence, $\Lambda \to 0$ when $n_e \to 0$.
High $\mathbf{n_e}$: At high n_e, collisional de–excitation dominates over radiative de–excitation. Thus, SII is in thermal equilibrium with the electrons, and $T_x \to T_e$ which is the kinetic temperature of the electrons. The cooling rate Λ reaches a finite value at high n_e. This is because the near–thermal–equilibrium state implies that collisions maintain a fixed ratio of $\frac{n_u}{n_l}$, as given by the Boltzmann equation. With this ratio and n_e fixed, the gas will keep emitting at a constant rate $\Lambda = n_3 A_{31} h\nu_{31}$. Note that here we are assuming $A_{32} << A_{31}$.

f) Calculate the flux (in $\mathbf{erg\,cm^{-2}\,s^{-1}}$) you would observe in the SII 6716 Å line from the Orion nebula. Assume $\mathbf{n_e = 1000\,cm^{-3}, T = 10^4\,K}$, a Strömgren radius of 1 pc, a distance of 500 pc, and an abundance ratio described by $\mathbf{n_{SII}/n_e = 10^{-4}}$.

The flux will be due to the energy lost by the gas in collisional de-excitation. Hence, for the 6716 Å line, we have:

$$\begin{aligned} L_c &= n_3 A_{31} h\nu_{31} \\ &= 2.1 \times 10^{-18}\,\mathrm{erg\,cm^{-3}\,s^{-1}} \end{aligned} \tag{24}$$

The total power emitted by the Strömgen sphere is obtained by multiplying this by the total volume. Finally, the flux is obtained by accounting for the dilution factor due to distance. Thus we have,

$$\begin{aligned} F &= \frac{L_c V}{4\pi d^2} \\ &= 2.1 \times 10^{-18} \cdot \frac{4\pi/3 \times (3.08 \times 10^{18})^3}{4\pi(500 \cdot 3.09 \times 10^{18})^2} \\ F &= 8.4 \times 10^{-6}\,\mathrm{erg\,cm^{-2}\,s^{-1}} \end{aligned} \tag{25}$$

3 Hubble Deep Field

In the Hubble Deep Field images, one of the most striking features is the high fraction of galaxies undergoing interactions or collisions. Suppose the random velocities of galaxies within clusters are typically $\mathbf{650\,km\,s^{-1}}$.

a) Calculate the temperature to which the ISM will be shock-heated in a galaxy-galaxy collision (assuming that the kinetic energy is thermalized in the interaction shock fronts).

The energies of the stars and the ISM are not very strongly coupled. So, only the kinetic energy of the ISM is thermalized. We can write:

$$\frac{1}{2} M_{ISM} v^2 = \frac{3}{2} \frac{M_{ISM}}{\mu m_H} kT \tag{26}$$

Table 5: Cooling timescales at different temperatures

ΔT	$\Lambda(\mathrm{erg\,cm^3\,s^{-1}})$	τ (s)	τ (years)
$10^7 \rightarrow 10^6$	2×10^{-23}	6.9×10^{13}	2200×10^3
$10^6 \rightarrow 10^5$	8×10^{-23}	1.8×10^{12}	57×10^3
$10^5 \rightarrow 10^4$	2×10^{-22}	6.9×10^{10}	2×10^3
$10^4 \rightarrow 10^3$	9×10^{-25}	1.5×10^{12}	48×10^3
$10^3 \rightarrow 10^2$	4×10^{-25}	3.5×10^{11}	11×10^3

where μ is the mean atomic weight of the gas, we assume $\mu = 0.7$. This gives the temperature of ISM to be:

$$T = \frac{\mu m_H v^2}{3k} = 1.2 \times 10^7\,\mathrm{K} \tag{27}$$

The high temperature implies that gas will be ionized, which justifies $\mu = 0.7$ instead of 1.4.

b) Assume that the ISM has mean density $1\,\mathrm{cm^{-3}}$; also assume that the gas is not compressed during the collision (not true in a full hydrodynamical treatment). Use the generalized ISM cooling curve shown below (for x = 0.1) to calculate the cooling times (required for gas to reduce its temperature by 50%) as the gas cools down through 10^6, 10^5, 10^4, 10^3, and 10^2 K. What would you conclude about the relative masses of gas at 10^6, 10^5, and 10^4 K?

A typical cooling timescale can be estimated by,

$$\tau \simeq \frac{kT}{\Lambda n} \tag{28}$$

Here, Λ represents the sum of all the cooling processes taking place in the gas. An exact calculation of cooling will require accounting for the compression of the gas as it cools, and integrating over the exact cooling rate $\Lambda(T)$ as a function of temperature. But in this simplified treatment we ignore these factors. The cooling timescales are calculated using Equation 28 and tabulated in Table 5. Note that in the range $10^5\,\mathrm{K} \rightarrow 10^4\,\mathrm{K}$ we have used a lower value of Λ than at $T = 10^5\,\mathrm{K}$. That is because the gas quickly cools from 10^5 K but spends greater time cooling at the lower cooling rate.

From the cooling curve and from Table 5, we see that the gas cools very quickly through 10^5 K. It spends much longer times at 10^6 K and 10^4 K – so we expect to find much higher quantities of gas at these temperatures. From these values we can say that:

$$M_{gas}(10^5\,\mathrm{K}) < M_{gas}(10^4\,\mathrm{K}) \simeq M_{gas}(10^6\,\mathrm{K}) \tag{29}$$