Ay102/Ay126 - ISM - Solutions Problem Set 3

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1. **[15 pts]** The first step in forming the molecular hydrogen will be create bonds between two hydrogen atoms and a dust grain. We might guess that the binding energy of a hydrogen atom to the dust grain is about the binding energy of a hydrogen molecule-dust grain system. This probably isn't exactly right, but is hopefully reasonably close. So the dust grain binding energies are unimportant compared to the H_2 binding energy (4.48 eV).

   As a result, the net heat input due to the formation of H_2 is just: the rate of H_2 formation × Energy that goes into the cloud as heat per each formed H_2. Putting numbers:

$$\begin{aligned} H_2 \text{ rate} &= 10^{-17} \text{ cm}^3 \text{ s}^{-1} \times (100 \text{ cm}^{-3})^2 \\ &= 10^{-13} \text{ cm}^{-3} \text{ s}^{-1} \\ Energy/H_2 &= \frac{2}{5} \times 4.48\text{eV} \\ &= 2.87 \times 10^{-12}\text{erg} \end{aligned}$$

   H_2 behaves as if it has five modes, so 2/5 of the energy is radiated away. As a consequence, the heating is $\Gamma \sim 2.87 \times 10^{-25}$ erg cm^3 s^{-1}. For a reasonably sized molecular cloud, this can be a solar luminosity or more.

2. **(a)[10 pts]** The flux from the star at a distance r is:

$$F_*(r,\lambda) = \pi B_\lambda(T_*)\frac{4\pi R_*^2}{4\pi r^2} = \pi B_\lambda(T_*)\frac{R_*^2}{r^2}$$

   And we know that the dust grain emits as a modified blackbody:

$$F_{dust}(\lambda) = \pi Q_{em}(\lambda) B_\lambda(T_{dust})$$

   At a distance r, the dust grain will absorb the radiation coming from the star as if it had a "disk-like" cross section:

$$\pi a^2 \int_0^\infty Q_{abs}(\lambda) F_*(r,\lambda) d\lambda$$

And then it will re-emit this radiations all over its surface:

$$4\pi a^2 \int_0^\infty F_{dust}(\lambda) d\lambda$$

If the dust grain is in radiative balance:

$$\pi a^2 \int_0^\infty Q_{abs}(\lambda) F_*(r,\lambda) d\lambda = 4\pi a^2 \int_0^\infty F_{dust}(\lambda) d\lambda$$

Hence:

$$\frac{R_*^2}{r^2}\int_0^\infty \frac{1}{\lambda^n} B_\lambda(T_*) d\lambda = 4\int_0^\infty \frac{1}{\lambda^n} B_\lambda(T_{dust}) d\lambda$$

From Tielens, page 127, we found the value of the integral to be:

$$\int_0^\infty \frac{1}{\lambda^n} B_\lambda(T) d\lambda = \frac{\sigma T^4}{\pi}\frac{15}{\pi^4}(n+3)!\zeta(n+4)\left(\frac{kT}{hc}\right)^n$$

where $\zeta(n)$ is the Riemann's function.

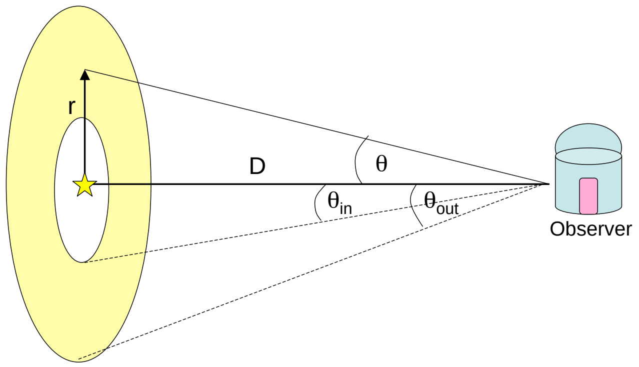
Eliminating the constans we obtain:

$$T_{dust} = T_*\left(\frac{R_*^2}{4r^2}\right)^{\frac{1}{4+n}}$$

(b)[20 pts] The flux at a distance $D = 10$ kpc coming from an optically thick disk is given by the integral:

$$\begin{aligned} F_\lambda &= \int_{4\pi} I_\lambda \cos\theta d\Omega \\ &= \int_{4\pi} B_\lambda(T_{dust}) \cos\theta d\Omega \\ &= 2\pi \int_{\theta_{in}}^{\theta_{out}} B_\lambda(T_{dust}) \cos\theta \sin\theta d\theta \end{aligned}$$

where we obtain a factor of 2π after integrating azimuthally. The geometry of the problem can be seen in the following figure.



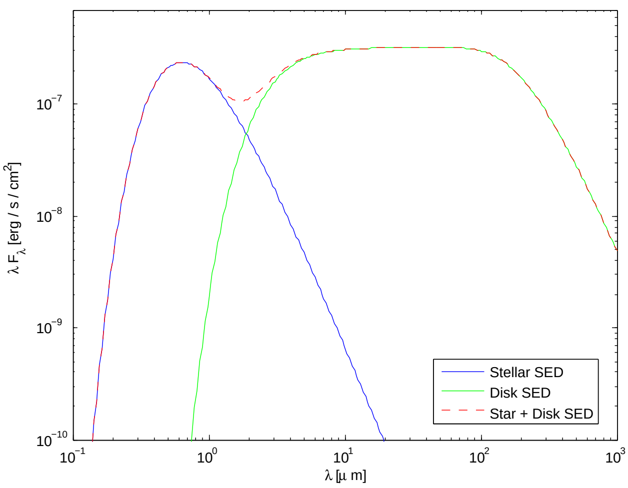
Changing variables $r \approx D \sin\theta$, $dr \approx D \cos\theta d\theta$:

$$\Rightarrow F_\lambda = 2\pi \int_{R_{in}}^{R_{out}} B_\lambda(T_{dust}(r)) \frac{rdr}{D^2}$$

For a given temperature profile ($T_{dust}(r)$), this would be the integral that we must solve numerically for each λ, in order to obtain the spectral energy distribution (λF_λ) of the disk. For the stellar SED this is much simpler, since the star is described by a single temperature, hence:

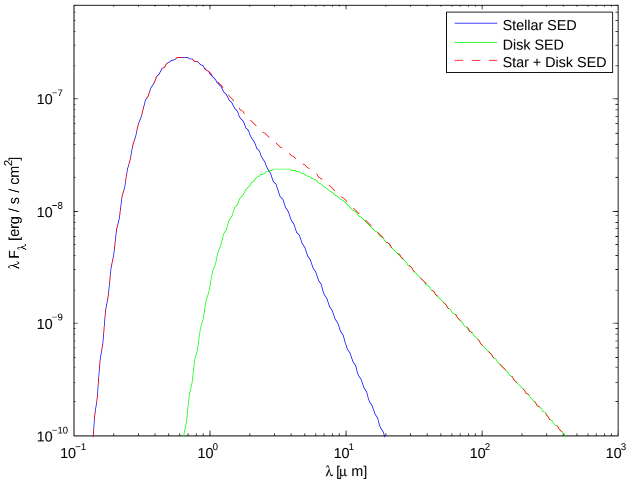
$$\begin{aligned} F_\lambda &= \int_{4\pi} I_\lambda \cos\theta d\Omega \\ &= \int_{4\pi} B_\lambda(T_*) \cos\theta d\Omega \\ &= \pi B_\lambda(T_*) \frac{R_*^2}{D^2} \end{aligned}$$

The SED for a star+ disk using the temperature profile obtained in part (a) is:

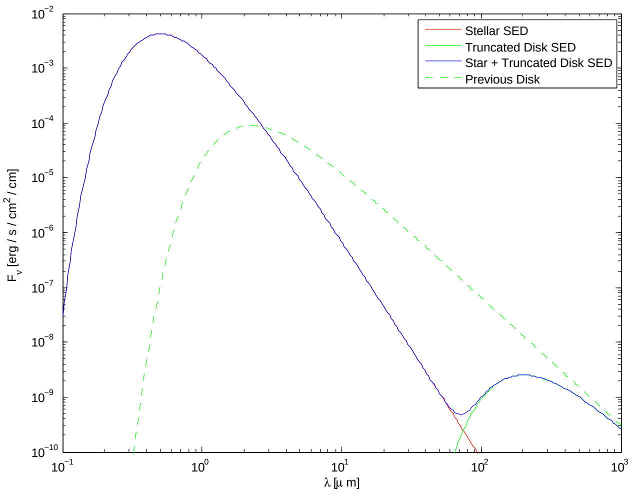


Note how the emission from the disk overwhelms the stellar emission at wavelengths larger than few μm.

Using the Chiang & Goldreich temperature profile, the SED looks like:



(c)[10 pts] As the disk evolves and clears out the inner parts of the disk we notice that the emission from the disk declines, and the peak moves to longer wavelengths as most of the contribution for its emission comes from large radii. Then to study such systems we have to go to sub-mm or mm wavelenghts.



3. **(a)[5 pts]** There are many ways to derive this length scale, one of them is to use the Virial theorem to calculate the size of a cloud that can balance its self-gravitational energy with its thermal energy:

$$2E_{thermal} + E_{grav} = 0$$
$$\Rightarrow 2 \times \frac{3}{2} NkT = \frac{3}{5} \frac{GM^2}{R}$$

where N is the total number of particles in the cloud: $N = \frac{M}{\mu m_H} = \frac{4\pi}{3} R^3 \frac{\rho}{\mu m_H}$.

A little algebra yields:

$$R_J = \left(\frac{15}{4\pi}\frac{kT}{G\rho\mu m_H}\right)^{1/2}$$

$$M_J = \left(\frac{5kT}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2}$$

(b)[5 pts] Assuming that all the other parameters remain unaffected, for a cloud smaller than R_J the gravitational and thermal energy follow that $|E_{grav}| > 2E_{thermal}$, and hence the cloud will be unbound.

On the other hand, if $|E_{grav}| < 2E_{thermal}$ (and hence $R > R_J$) the cloud is bound. Since the gravitational energy scales like R^5, while the thermal energy scales like R^3, the gravitational force will increase more rapidly than the thermal force, and we can have collapse.

(c)[5 pts] For T=15 K and $n = 100$ cm^{-3}, the Jean mass is: $M_J \sim 380 M_\odot$ (this result might vary according to the approximations you used when finding the expression for M_J).

(d)[5 pts] Observationally, we know that there are stars with mass much less that our limit calculated in (c). The physical process that we think leads to the formation of low-mass stars in dense cores of GMCs is fragmentation. Any inhomogeneity in the density will cause that location to satisfy the Jean mass limit and collapse independently of the rest of the core, and hence fragmentation will occur forming multitude of smaller objects. But there are some unresolved issues, like: if this reasoning is correct the star formation efficiency will be quite large, but observations show that about 1% of the cloud actually form stars, and in this fragmentation scenario how do we form large stars? Although a GMC has a mass larger its Jean mass limit it would not collapse because there are other ways to support it against gravitation (like magnetic fields, rotation, turbulence, etc.) besides just thermal pressure.

(e)[10 pts] i.- For an isothermal collapse: $M_J \propto \rho^{-1/2}$. For an adiabatic collapse: $T \propto \rho^{\gamma-1}$, then $M_J \propto \rho^{(3\gamma-4)/2}$.

ii.- The gravitational potential energy of a self-gravitating object is defined as the negative of the amount of energy that is required to disperse its mass to infinity, and can be written as:

$$\Omega = \int_0^M -\frac{GM_r}{r}dM_r$$

If we assume that the density is constant throughout the cloud, the previous integral is calculated as:

$$\Omega = \int_0^M -\frac{G}{r}\left(\frac{4\pi}{3}r^3\rho\right)4\pi r^2\rho dr = -\frac{3}{5}\frac{GM^2}{R}$$

Free-fall timescale: The equation of motion for a test particle of mass m at a distance r of the center of the cloud of mass M and radius R is:

$$m\ddot{r} = \frac{-GM_r m}{r^2}$$

Assuming a constant density $\bar{\rho}$ throughout the cloud we can write: $M_r = \frac{4\pi}{3}r^3\bar{\rho}$, replacing this in the previous equation gives a S.H.O.:

$$\ddot{r} + \frac{4\pi G\bar{\rho}}{3}r = 0$$

with a characteristic frequency of oscillation $\omega = \sqrt{\frac{4\pi}{3}G\bar{\rho}}$. The typical time scale of the particle to fall into the gravitational potential of the cloud will be given by the inverse of the frequency of oscillation:

$$t_{ff} = \sqrt{\frac{3}{4\pi G\bar{\rho}}} \sim \frac{1}{\sqrt{G\bar{\rho}}}$$

iii.- Since we are at the transition point between adiabatic and isothermal collapse we will have that the cloud's luminosity will be equal to the "gravitational" luminosity and to the "radiative" luminosity:

$$L_{cloud} = \frac{\Delta E_{grav}}{t_{ff}} = eL_{rad}$$

where:

$$L_{rad} = 4\pi R^2\sigma T^4$$

$$L_{grav} = \frac{3}{5}G^{3/2}\left(\frac{M}{R}\right)^{5/2}$$

Replacing the radius as a function of mass and density, and using the expression for the Jean mass as a function of T and ρ we obtain:

$$M_{J,min} = 0.03\frac{T^{1/4}}{e^{1/2}\mu^{9/4}}M_\odot$$

Evaluating for typical values: $M_{J,min} \sim 0.5 M_\odot$.

4. **(a)[5 pts]** $R = 5.5$ kpc, $d_1 = 10.9$ kpc, $d_2 = 3.9$ kpc.

(b)[5 pts] The virial theorem states: $2\langle T \rangle + \langle U \rangle = 0$. For a spherical cloud of mass M, radius R and velocity dispersion in the line of sight σ_{los}, we have:

$$\langle U \rangle = -\frac{3}{5}\frac{GM^2}{R}$$
$$\langle T \rangle = -\frac{3}{2}M\sigma_{los}^2$$

Solving for the mass we find:

$$M_{virial} = \frac{5R\sigma_{los}^2}{G} = \frac{5R(1.36v_{FWHM})^2}{G}$$

Using the values given for the cloud's angular diameter and FWHM for the CO line we have that: for $d_1 = 10.9$ kpc, $M_{vir} \sim 4.9 \times 10^4 M_\odot$, for $d_2 = 3.9$ kpc, $M_{vir} \sim 1.8 \times 10^4 M_\odot$.

(c)[5 pts] Since the CO(1-0) line is optically thick, the detailed balance equation from where we calculate the critical density is not only given by spontaneous decay, but we also have to include the stimulated emission and absorption terms because the radiation has a hard time to escape:

$$n_u A_{ul} + n_u B_{ul} J = n_l B_{lu} J + n_l n_e q_{ul}$$

Since $\tau \gg 1$ there is a photon escape probability β in the radiative field J:

$$J = (1-\beta)\frac{4\pi}{c}B_\nu(T)$$

where β is defined as: $\beta = \frac{1}{\tau}(1 - e^{-\tau})$, so for optically thin material $\beta \to 1$. Replacing all the above in the detailed balance equation, and using the Einstein relations for the Einstein coefficients:

$$\frac{n_u}{n_l}A_{ul}\left[1 + \frac{1-\beta}{e^{h\nu/kT} - 1}\right] = \frac{g_u}{g_l}\frac{A_{ul}(1-\beta)}{e^{h\nu/kT} - 1} + n_e q_{ul}$$

which can be solved to give: $n_{e,crit} = \beta\frac{A_{ul}}{q_{ul}}$.

To estimate the collisional coefficient q_{ul}, we remember that this is just an average of the particle cross-section and velocity, hence: $q_{ul} \sim \langle \sigma v \rangle$. For CO $\sigma \sim 10^{-15}$ cm^2, and we calculate the velocity from $v \sim \sqrt{3kT/m} \sim 1.4 \times 10^4$ cm/s. So: $n_{e,crit} \sim 85.7$ cm^{-3}.

(d)[5 pts] The mean density for each distance is:

$$n = M_{virial}/(\frac{4\pi}{3}R^3 m_{H_2}) = 65.5\text{cm}^{-3}, 507\text{cm}^{-3}$$

And the excitation temperature:

$$T_{ex} = \frac{\Delta E_{ul}/k}{\Delta E_{ul}/kT + \ln(1 + n_{crit}/n)} = 5.0K, 13.0K$$

(e)[5 pts] The brightness temperature (T_B) of the CO line is related with the background temperature (T_{CMB}) and the excitation temperature by the radiative transport equation:

$$T_B = T_{CMB}e^{-\tau} + T_{ex}(1 - e^{-\tau})$$

Since τ is much larger than 1, $T_B \to T_{ex}$