Ay102/Ay126 - ISM - Solutions Problem Set 5

Solutions by Andrew Baker, Roy Gal, Mansi Kasliwal and Laura M. Pérez

1. (a) [20 pts] The kinetic energy is $E = \frac{1}{2}Mu^2$, where $M = M(t)$ is the mass of gas in the swept-up shell and $u = \frac{dr}{dt}$ is the velocity of the blast wave. We can compute $M(t)$ using the given density distribution:

$$\begin{aligned} M(t) &= \int_0^r 4\pi r^2 \rho(r) dr \\ &= 4\pi\rho_0 r_0^\beta \int_0^r r^{2-\beta} dr = \frac{4\pi\rho_0 r_0^\beta}{3-\beta} r^{3-\beta} \end{aligned}$$

Now use the power-law parametrization for $E(t)$ to obtain

$$E_0 t^\alpha = \frac{2\pi\rho_0 r_0^\beta}{3-\beta} r^{3-\beta} \Big(\frac{dr}{dt}\Big)^2$$

Taking the square root of both sides and separating variables gives

$$\Big(\frac{3-\beta}{2\pi}\frac{E_0}{\rho_0 r_0^\beta}\Big)^{\frac{1}{2}} t^{\frac{\alpha}{2}} dt = r^{\frac{3-\beta}{2}} dr$$

Integrating we find:

$$\frac{2}{2+\alpha}\Big(\frac{3-\beta}{2\pi}\frac{E_0}{\rho_0 r_0^\beta}\Big)^{\frac{1}{2}} t^{\frac{2+\alpha}{2}} = \frac{2}{5-\beta} r^{\frac{5-\beta}{2}} + K$$

where K is a constant of integration. Upon solving for $r(t)$, K corresponds to $r(t=0)$, so if we take $r(t=0) = 0$, we have:

$$r(t) = \Big(\frac{5-\beta}{2+\alpha}\Big)^{\frac{2}{5-\beta}} \Big(\frac{3-\beta}{2\pi}\frac{E_0}{\rho_0 r_0^\beta}\Big)^{\frac{1}{5-\beta}} t^{\frac{2+\alpha}{5-\beta}}$$

Note that for the case $\alpha = \beta = 0$, we recover the Taylor-Sedov solution derived in class for a constant-energy blast wave expanding into a uniform-density medium.

(b) [5 pts] For the case $\alpha = 0$, the dependence of $r(t)$ on β is:

$$r(t) \propto (5-\beta)^{\frac{2}{5-\beta}} (3-\beta)^{\frac{1}{5-\beta}} t^{\frac{2}{5-\beta}}$$

Differentiating twice gives the acceleration:

$$\frac{d^2 r(t)}{dt^2} \propto -(5-\beta)^{\frac{2\beta-8}{5-\beta}} (3-\beta)^{\frac{6-\beta}{5-\beta}} t^{\frac{2\beta-8}{5-\beta}}$$

For the blast wave to be Rayleigh-Taylor stable, we require that it be decelerating:

$$-(5-\beta)^{\frac{2\beta-8}{5-\beta}} (3-\beta)^{\frac{6-\beta}{5-\beta}} < 0$$

and thus $\beta < 3$. For the blast wave to be Rayleigh-Taylor unstable, $\beta > 3$. Note that this critical exponent would be $\beta_{crit} = 2$ if the blast wave were being driven by a stellar wind, since in this case luminosity L rather than E would be nearly constant.

2. (a) [10 pts] In steady state, the reaction rates are equal: $J n_{H_2} = \alpha_3 n_{H_3^+} n_e$. Assuming overall charge neutrality, $n_e \sim n_{H_3^+}$. Thus,

$$\frac{n_e}{n_{H_2}} \sim \frac{J}{\alpha_3\ n_e}$$

$$\Rightarrow \frac{n_e}{n_{H_2}} \sim \sqrt{\frac{J}{\alpha_3 n_{H_2}}}$$

(b) [5 pts] The free-fall timescale is given by:

$$\begin{aligned} t_{ff} &= \sqrt{\frac{3}{4\pi G\rho}} \sim \sqrt{\frac{3}{4\pi G(m_{H_2} n_{H_2})}} \\ &\sim 3\times 10^7\ n_{H_2}^{-0.5}\ \text{yrs} \end{aligned}$$

The ambipolar difussion timescale is: (from Eq. 13.57, *Physical Processes in the Interstellar Medium*, Spitzer L.)

$$t_{AD} = \frac{n_i\langle\sigma v\rangle}{2\pi G m_H n_H} \sim 5\times 10^{13}\frac{n_i}{n_H}\ \text{years}$$

Since $n_i = n_e$, and $\alpha_3 \sim 4\times 10^{-6}$ cm^3 s^{-1} (from Stahler & Palla, *The Formation of Stars*, page 219), we obtain:

$$\begin{aligned} t_{AD} &\sim \Big(\frac{5\times 10^{13}}{\text{years}}\Big)\Big(\frac{n_{H_2}}{n_H}\Big)\sqrt{\frac{J}{\alpha_3 n_{H_2}}} \\ &\sim \Big(\frac{8\times 10^{7}}{\text{years}}\Big)\Big(\frac{n_{H_2}}{n_H}\Big)\sqrt{\frac{J}{10^{-17}\text{s}^{-1}}} n_{H_2}^{-0.5} \end{aligned}$$

Thus,

$$\frac{t_{AD}}{t_{ff}} \approx 3\left(\frac{n_{H_2}}{n_H}\right)\sqrt{\frac{J}{10^{-17}\mathrm{s}^{-1}}}$$

3. We start with the MHD equations:

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}P - \rho\vec{\nabla}\phi + \frac{1}{8\pi}(\vec{\nabla}\times\vec{B})\times\vec{B} \quad (1)$$

$$\vec{\nabla}\times(\vec{v}\times\vec{B}) = \frac{\partial\vec{B}}{\partial t} - \frac{1}{4\pi\sigma}\nabla^2\vec{B} \quad (2)$$

Together with Maxwell's equation: $\nabla\cdot\vec{B} = 0$.

Assuming high conductivity ($\sigma\to\infty$), that the magnetic field only components are $\vec{B} = B_\theta\hat{\theta} + B_z\hat{z}$ and that $\vec{v} = v_\theta\hat{\theta}$ we can rewrite equation (1) as:

$$\rho\frac{d\vec{v}}{dt} = -\frac{1}{4\pi}\vec{B}\times(\vec{\times}\vec{B})$$

where:

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{\partial\vec{v}}{\partial t}$$

since $(\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{1}{r}v_\theta\frac{\partial v_\theta}{\partial\theta}\hat{\theta} = 0$

Now, is easy to show that given the constraints on $\vec{B}$:

$$[\vec{B}\times(\vec{\nabla}\times\vec{B})]_\theta = -B_z\frac{\partial B_\theta}{\partial z}$$

Thus:

$$\frac{dv_\theta}{dt} = \frac{1}{4\pi\rho}B_z\frac{\partial B_\theta}{\partial z} \quad (3)$$

Now, each term of equation (2) can be written as: (taking only the $\hat{\theta}$ component)

$$\vec{v}\times\vec{B} = v_\theta B_z\hat{r}$$

$$\Rightarrow [\vec{\nabla}\times(\vec{v}\times\vec{B})]_\theta = \frac{\partial}{\partial z}(v_\theta B_z)\hat{\theta}$$

$$\nabla^2\vec{B} = 2\vec{B}\times(\vec{\nabla}\times\vec{B}) + 2(\vec{B}\cdot\vec{\nabla})\vec{B}$$

where:

$$[(\vec{B}\cdot\vec{\nabla})\vec{B}]_\theta = B_z\frac{\partial B_\theta}{\partial z}\hat{\theta}$$

$$\Rightarrow \nabla^2\vec{B} = -2B_z\frac{\partial B_\theta}{\partial z}\hat{\theta} + 2B_z\frac{\partial B_\theta}{\partial z}\hat{\theta} = 0$$

So equation (2) is now:

$$\frac{\partial B_\theta}{\partial t} = B_z\frac{\partial v_\theta}{\partial z} \quad (4)$$

Putting equation (3) and (4) together yields two wave equations:

$$\frac{\partial^2 B_\theta}{\partial t^2} = \frac{B_z^2}{4\pi\rho}\frac{\partial^2 B_\theta}{\partial z^2} = v_{\mathrm{Alfven}}^2\frac{\partial^2 B_\theta}{\partial z^2}$$

$$\frac{\partial^2 v_\theta}{\partial t^2} = \frac{B_z^2}{4\pi\rho}\frac{\partial^2 v_\theta}{\partial z^2} = v_{\mathrm{Alfven}}^2\frac{\partial^2 v_\theta}{\partial z^2}$$

The general solution is:

$$v_\theta(r,t) = \underbrace{f\left(t - \frac{z}{v_{\mathrm{Alfven}}}, r\right)}_{\text{Retarded part}} + \underbrace{g\left(t + \frac{z}{v_{\mathrm{Alfven}}}, r\right)}_{\text{Advanced part}}$$

Because the perturbation can't disturb the medium at distances greater than the ellapsed time $\times$ the velocity of the wave, the advanced part of the wave must be zero for disturbances along $+z$, while the retarded part of the wave must be zero for disturbances along $-z$.

As the waves propagate through the plasma they set into rotation about $\vec{B}$ with velocity v_θ, so for an element of material located at (r,z) on time t, its angular momentum would be:

$$\rho r\times v\left(t - \frac{z}{v_{\mathrm{A}}}, r\right)$$

So the total angular momentum is given by the integral of all contributions where the medium has been perturbed, that is at a time t and hence a distace of $d = v_{\mathrm{Alfven}}t$:

$$\begin{aligned} L_{cloud} &= \int_{Vol}\rho r v\left(t - \frac{z}{v_{\mathrm{A}}}, r\right)dV \\ &= 4\pi d\times\int_0^{d=v_{\mathrm{A}}t}\rho r v\left(t - \frac{z}{v_{\mathrm{A}}}, r\right)r dr \end{aligned}$$

Assuming $v\left(t - \frac{z}{v_{\mathrm{A}}}, r\right) = \omega(t)r$:

$$\Rightarrow L_{cloud} = 4\pi v_{\mathrm{A}}t\rho\omega(t)\int_0^d r^3 dr$$

So the angular momentum loss becomes:

$$\frac{dL_{cloud}}{dt} = -4\pi v_{\mathrm{A}}\omega(t)\int_0^d\rho r^3 dr$$

(b) Given a mass and density for the condensation ($M = 10M_\odot$, $\rho = 4\times10^{-17}$ gm cm^{-3}), its radius is:

$$R = \left(\frac{3M}{4\pi\rho}\right)^{1/3} = 4.9\times10^{16}\text{ cm} = 0.016\text{ pc}$$

A typical value for the angular velocity of a main sequence star is $\omega_{MS}\sim 2\times10^{-6}$ Hz. The radius for a 10 $M_\odot$ star on the main sequence is

$R_{MS} \sim 6R_\odot$. The idea is to calculate what would have been the angular velocity of the condensation (ω_{Clump}) if the angular momentum is conserved all the way to the main sequence stage:

$$\Rightarrow (M\omega R^2)_{Clump} = (M\omega_{MS}(6R_\odot)^2)_{MS}$$

$$\Rightarrow \omega_{Clump} = \omega_{MS}\left(\frac{6R_\odot}{R_{initial}}\right)^2 \sim 2\times 10^{-16}\text{Hz}$$

Now, coming back to the equation of angular momentum losses, we need to calculate the angular velocity as a function of time, so we assume that at any time the angular momentum can be expressed as: $L_{cloud} = \frac{2}{5}MR^2\omega(t)$, and furthermore assuming that the mass, radius, and hence density is conserved (the condensation remains of the same size, and the angular momentum is loss through Alfven waves) we obtain:

$$\frac{dL_{cloud}}{dt} = -4\pi v_{\rm A}\omega(t)\int_0^R \rho r^3 dr$$

$$\Rightarrow -\pi v_A \rho\omega(t)R^4 = \frac{2}{5}MR^2\frac{d\omega(t)}{dt}$$

Solving for $\omega(t)$ we found that: $\omega(t) = \omega(0)e^{-t/\tau}$

$$\text{with } \tau = \frac{4M}{5\sqrt{\pi\rho}BR^2} = 6\times 10^6 \text{ years}$$

So the time required for the condensation to slow down from $\omega_0 = 2\times 10^{-15}$ Hz to ω_{Clump} is:

$$t = \tau \ln\left(\frac{\omega_0}{\omega_{Clump}}\right) \sim 1.4\times 10^7 \text{ yrs}$$

Compared with the free-fall timescale, calculated when the condensation had a radius of $R = 0.1$ pc and hence a density of $\rho = 1.6\times 19^{-19}$ gm cm^{-3}, which gives a timescale of $t_{ff} \sim 1/\sqrt{G\rho} \sim 3\times 10^5$ yrs, we see that the collapse occurs before angular momentum losses.

4. (a) The mean free path of each component of the ISM is given by:

$$l = \frac{1}{n\sigma} = \frac{1}{n\pi R^2}$$

where n is the average number density of each component (WIM or CNM) and σ is the cross-section of the cloud, that we assume to be spherical. Then for each component the m.f.p is: $l_{WIM} = 42.4$ pc, $l_{CNM} = 169.8$ pc. Then, to calculate the number of clouds intercepted along the line of sight to a star 1 kpc away in the disk we just divide the m.f.p by the distance to the star:

$$N_{WIM} = \frac{1\times 10^3 \text{ pc}}{42.4 \text{ pc}} \sim 24$$

$$N_{CNM} = \frac{1\times 10^3 \text{ pc}}{169.8 \text{ pc}} \sim 6$$

(b) The dispersion measure is $DM = \int_0^L n_e dl$. The CNM doesn't contribute with electrons since is a neutral material, while the WIM and the coronal gas (CG) are mostly ionized, so it is fair to assume that $n_e \sim n_H$. We also assume that the density of electrons is constant across each component, so the dispersion measure is simply:

$$DM = n_{e,WIM}\times L_{WIM} + n_{e,CG}\times L_{CG}$$

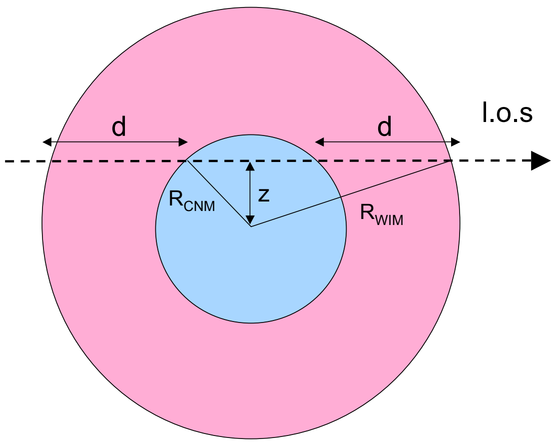
where L_{WIM} and L_{CG} are the distances traveled along the line of sight throughout the WIM and coronal gas, hence:

$$L_{WIM} = N_{WIM}\times\langle d_{WIM}\rangle$$

$$L_{CG} = 1\text{ kpc} - N_{WIM}\times\langle d_{WIM+CNM}\rangle$$

with $\langle d_{WIM}\rangle$ the average distance that you travel throughout a single WIM cloud and $\langle d_{WIM+CNM}\rangle$ the average distance that you travel throughout a cloud with both CNM and WIM.

To calculate those average distances we have to consider the geometry of the problem:



In the figure, d is the actual length of the region that is going to contribute with electrons to the emission measure, hence: $d = 2(\sqrt{R_{WIM} - z^2} -$

$\sqrt{R_{CNM} - z^2}$), and z is the distance from the center of the cloud to the line of sight (l.o.s.). Note that when $z > R_{CNM}$ then d is just $2\sqrt{R_{WIM} - z^2}$.

Now, every l.o.s is going to have the same probability, but this doesn't mean that every z is going to be equally probable, since the regions further from the center are more probable to be crossed with a l.o.s. (remember that there is no angular dependance). So the probability weight function to calculate the average distance traveled throughout a single cloud is just $2\pi z dz$, hence:

$$\langle d_{WIM} \rangle = \int_0^{R_W} d(z) 2\pi z dz$$

And separating this integral between $z < R_C$ and $z > R_C$ we obtain:

$$\begin{aligned}
\langle d_W \rangle &= 4\pi \int_0^{R_C} [\sqrt{R_W - z^2} - \sqrt{R_C - z^2}] z dz \\
&\quad + \int_{R_C}^{R_W} 2\sqrt{R_W - z^2} 2\pi z dz \\
&= \frac{4}{3} \left(\frac{R_W^3 - R_C^3}{R_W^2} \right)
\end{aligned}$$

Is easy to see that $\langle d_{WIM+CNM} \rangle$ is just the same calculation as above but for a whole sphere, that is taking the last result but with $R_{CNM} = 0$:

$$\langle d_{WIM+CNM} \rangle = \frac{4}{3} R_{WIM}$$

So the dispersion measure:

$$\begin{aligned}
DM &= n_{e,W} N_W \frac{4}{3} \left(\frac{R_W^3 - R_C^3}{R_W^2} \right) + \\
&\quad + n_{e,CG} \left(1 \text{ kpc} - N_W \frac{4}{3} R_W \right) \\
&= 42 \text{ cm}^{-3} \text{ pc} + 2.52 \text{ cm}^{-3} \text{ pc} \\
&= 44.5 \text{ cm}^{-3} \text{ pc}
\end{aligned}$$

(c) Looking along a line of sight leaving the Galaxy in the vertical direction, there would be contributions to the emission measure only from 0.5 kpc, assuming we are sitting in the middle of the disk, so the number of clouds that we intercept is half what we obtain before $N_{WIM} \sim 12$. The procedure is the same as above:

$$\begin{aligned}
EM &= \int_0^L n_e^2 dl \\
&= n_{e,WIM}^2 \times L_{WIM} + n_{e,CG}^2 \times L_{CG} \\
&= n_{e,W}^2 N_W \frac{4}{3} \left(\frac{R_W^3 - R_C^3}{R_W^2} \right) + \\
&\quad + n_{e,CG}^2 \left(0.5 \text{ kpc} - N_W \frac{4}{3} R_W \right) \\
&= 6.3 \text{ cm}^{-6} \text{ pc} + 0.004 \text{ cm}^{-6} \text{ pc} \\
&= 6.304 \text{ cm}^{-6} \text{ pc}
\end{aligned}$$

(d) The mean gas density would be:

$$\langle n_H \rangle = n_{H,W} V_W n_W + n_{H,C} V_C n_C + n_{H,CG}$$

where V_W, V_C are the volume of one single WIM or CNM cloud. Putting numbers we obtain:

$$\begin{aligned}
\langle n_H \rangle &= [4.1 \times 10^{-2} + 0.58 + 3 \times 10^{-3}] \text{ cm}^{-3} \\
&= 0.63 \text{ cm}^{-3}
\end{aligned}$$

The fraction of the total volume that is occupied by each phase is given by the volume of one cloud (V_{WIM}, V_{CNM}) times the number density of clouds in that particular phase ($n_{WIM} = n_{CNM} = 3 \times 10^{-4}$ pc^{-3}) :

$$\begin{aligned}
f_{WIM} = V_{WIM} \times n_{WIM} = 0.14 \\
f_{CNM} = V_{CNM} \times n_{CNM} = 0.02 \\
f_{CG} = 1 - f_{WIM} - f_{CNM} = 0.84
\end{aligned}$$