Ay 20 Basic Astronomy and the Galaxy
Problem Set 5

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December 7, 2007

1 One electron atoms

a) Orbital radii and energies First, let us write down the equations we know regarding the one-electron atom. The potential energy of the electron is given by Coulomb's law:

$$E_{pot} = \frac{-Ze^2}{r} \tag{1}$$

in CGS units, where Z is the atomic number of the atom, e is the electronic charge, and r is the radius of the orbit. The radius of the orbit is such that the Coulomb attraction provides the centripetal force, hence if the mass of the electron is m and its velocity is v, we have:

$$\frac{mv^2}{r} = \frac{Ze^2}{r^2} \tag{2}$$

$$v = \sqrt{\frac{Ze^2}{mr}} \tag{3}$$

This gives the total energy of the electron:

$$E = \frac{-Ze^2}{r} + \frac{1}{2}mv^2 = \frac{-Ze^2}{2r} \tag{4}$$

Now we add the quantization condition:

$$L = mvr = n\hbar \tag{5}$$

$$m\sqrt{\frac{Ze^2}{mr}} = n\hbar \tag{6}$$

This gives us the radius of the nth orbital:

$$r_n = \frac{n^2\hbar^2}{Ze^2m} \tag{7}$$

Substituting the radius in Equation 4 gives us an experssion for the energy of an electron in the nth orbital:

$$E_n = \frac{-Ze^2}{2r} = \frac{-1}{n^2}\frac{Z^2me^4}{2\hbar^2} \tag{8}$$

The corresponding expressions in SI units are:

$$r_n = 4\pi\epsilon_0 \frac{n^2\hbar^2}{Ze^2m} \tag{9}$$

$$E_n = \frac{-1}{n^2}\frac{Z^2me^4}{2\hbar^2} \times \frac{1}{(4\pi\epsilon_0)^2} \tag{10}$$

b) He II atom To calculate the radius of the ground state orbit of singly ionized helium, we simply substitute the values into Equation 7. Note that if you use the CGS equation, then $e = 4.803 \times 10^{-10}$ e.s.u. Hence we get:

$$r_1(\text{He II}) = 0.26\,\text{Å} \tag{11}$$

The ground–state energy is given by Equation 8:

$$E_1 = -54.4\,\text{eV} = 8.7 \times 10^{-11}\,\text{erg} \tag{12}$$

The ionization energy is simply the negative of the ground state energy:

$$\chi_I = -E_1 = +54.4\,\text{eV} \tag{13}$$

c) Energy differences in isotopes In deriving the formulae above, we have assumed that the nucleus is lot more massive as compared to the electron. Hence, in Equations 2 and 5 we have used m, the mass of the electron. In an exact solution, m has to be replaced with μ, the reduced mass of the electron and nucleus. If the mass of the nucleus is m_N, mass of the electron is m, then the reduced mass is given by,

$$\mu = \frac{m_N m}{m_N + m} \tag{14}$$

Thus the m in Equation 8 gets replaced by μ. The ratio of μ for ${}^4_2$He II and ${}^3_2$He II is 1.000045, leading to very small differences in energy of different levels. These energy differences cause subtle differences in the spectra of these elements. The are very hard to measure, but are used to probe isotopic abundances of elements in astronomical systems.

2 The 51 Pegasi system

a) Mass Refering to Figure 7.7 in C&O, we can see that the points on the log–log plot of L versus M is a straight line - thus mass and luminoisty must be related by a power law. We can fit a straight line to it by reading off any two points. We see that (log M, log L) = (-0.75, -3) and (1.25, 5) are points on the line. Also, the the line passes through (0,0). This tells us that if we measure L in units of solar luminosity ($L_\odot$) and mass in units of solar masses $M_\odot$ then we have:

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^4 \tag{15}$$

Substituting L=1.3 $L_\odot$ gives M=1.1 $M_\odot$.

b) Radius To calculate the radius, we use the luminosity and effective temperature. We have,

$$L = 4\pi R_*^2 \sigma T_*^4 \tag{16}$$

Substituting $L_\odot = 3.8 \times 10^{33}\,\mathrm{erg\,s^{-1}}$, $T_* = 5660\,\mathrm{K}$, we get:

$$R_* = 8.3 \times 10^{10}\,\mathrm{cm} = 1.2\,R_\odot \tag{17}$$

c) Variation of Hβ wavelength The wavelength of the Hβ line measured in air is λ_0 4861.34 Å. We see from Figure 1 in the problem set that v_r varies from -60 m/s to +55 m/s. We can then calculate the variation in range of variation of λ by using the formula,

$$\Delta\lambda = \frac{v}{c}\lambda \tag{18}$$

Hence λ varies from $\lambda_0 - 9.7 \times 10^{-4}$ Å to $\lambda_0 + 8.9 \times 10^{-4}$ Å, corresponding to a total variation of 0.19 pm.

d) Selecting a good line The full width at half maximum (FWHM) of the Hβ line in the solar spectrum is about 0.1 Å. To measure velocities of $\sim$5 m/s, we have to measure $\Delta\lambda \sim 10^{-4}$ Å. Hence we have to centroid the line to 10^{-4} times its width, which will be prone to a lot of errors. A better choice will be to use one of the narrower lines in the spectrum, which are much sharper. Note that the Hβ line is broad and hence is the main contributor to the signal - resolving narrow lines requires high resolution spectroscopy and a high signal–to–noise ratio.

e) The planet

i. Semi major axis We use Kepler's law to calculate the semimajor axis of the orbit:

$$\frac{P^2}{a^3} = \frac{1}{M_* + M_p} \tag{19}$$

where P is the period in years, a is the semimajor axis in AU, M_* is mass of the star, M_p is the mass of the planet, both measured in solar masses. Neglecting the mass of the planet, we get:

$$a = \sqrt[3]{1.1 \times \left(\frac{4.231}{365.25}\right)^2} = 0.053\,\mathrm{AU} = 7.9 \times 10^{11}\,\mathrm{cm} \tag{20}$$

ii. Mass of the planet To estimate the mass of the planet, we use the modulations in the star's velocity. We know that:

$$M_* v_* = M_p v_p \tag{21}$$

$$v_p = \frac{2\pi a}{P} = 1.4 \times 10^7\,\mathrm{cm\,s^{-1}} \tag{22}$$

Substituting this value in Equation 21 we get,

$$M_p = 1.1\,M_\odot \times \frac{(55 - (-60)) \times 100}{2}\,\mathrm{cm\,s^{-1}} \times \frac{1}{1.4 \times 10^7\,\mathrm{cm\,s^{-1}}} = 4.7 \times 10^{-4}\,M_\odot = 9.3 \times 10^{29}\,\mathrm{gm} \tag{23}$$

f) Temperature of the planet We assume that the planet rotates rapidly and the surface is at a uniform temperature. The flux from the star at the planet's orbit is given by,

$$F = \frac{L_*}{4\pi a^2} \tag{24}$$

The energy absorbed by the planet is then:

$$E_{in} = 0.5 \times \pi R_p^2 F = \frac{\pi R_p^2 L_*}{8\pi a^2} \tag{25}$$

This energy is re–radiated as a blackbody:

$$E_{out} = 4\pi R_p^2 \sigma T_p^4 \tag{26}$$

Solving Equations 25 and 26 together we get:

$$T_p = \sqrt[4]{\frac{L_*}{4\pi a^2} \times \frac{\pi R_p^2}{2} \times \frac{1}{4\pi R_P^2 \sigma}} = 1100\,\mathrm{K} \tag{27}$$

In reality, the planet is orbiting so close to the star, that it is "tidally locked" to the star, just as moon is locked to the earth. So, the same surface of the planet faces the star all the time. At a place on the planet which has high noon, the starlight is incident normally on the "ground". The temperature can then be calculated by equating the energy absorbed and re–radiated by a unit area:

$$0.5 \times \frac{L_*}{4\pi a^2} = \sigma T^4 \tag{28}$$

This gives a temperature $\sqrt{2}$ times higher than the previous case: $T_p \sim 1550\,\mathrm{K}$

3 Finding Exoplanets

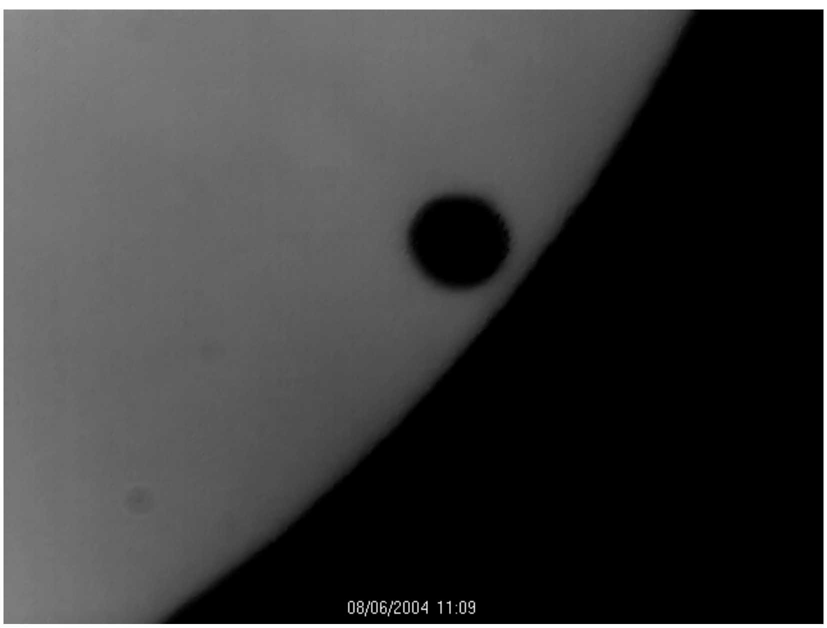

Figure 1: Image of venus transiting the sun, 8 June 2004. (Webcam on 6 inch newtonian telescope with solar filter)

a) Detecting Jupiter Since we are searching for a planet around a distant star, we can consider parallel rays coming from the star. If we could resolve the disc of the star, we would see the transit simply as a dark dot on the stars disc. (For example see Figure 1 for an image of venus transiting the disc of the sun.) For a distant star, we cacnt resolve the disc, so we will just note a drop in the flux of the star. The fractional decrease is simply given by the ratio of the areas:

$$f = \frac{\pi R_*^2}{\pi R_J^2} = 0.01 \tag{29}$$

Where we have used $R_J = 7.14 \times 10^9\,\text{cm}$. A useful website to get such constants is http://www.astro.wisc.edu/~dolan/constants.html. Here, we have also neglected any emission from Jupiter itself. If jupiter was also radiating significantly, there would be another correction to this, given by:

$$\Delta f = \frac{T_J^4}{T_\odot^4} \tag{30}$$

Let us again assume an albedo of 0.5. We can either estimate T_J or we can directly use Equation 27, substituting $L_* = 4\pi R_\odot^2 \sigma T_\odot^4$, to get:

$$\frac{T_J^4}{T_\odot^4} = \frac{R_\odot^2}{8a^2} \approx 10^{-7} \tag{31}$$

Hence, we can ignore this correction in calculating the fractional decrease in brightness.

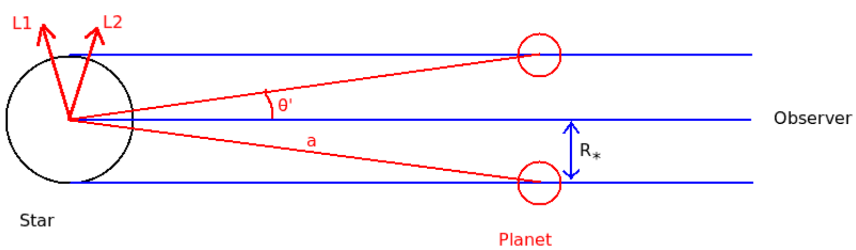

Figure 2: Range of angles in which an observer will see a transit/occultation

b) Probability of finding Jupiter Refer to Figure 2. The planet will occult the star only if the angular momentum vector is in the range L1 to L2, which means $\theta' < sin^{-1}\left(\frac{R_\odot}{a}\right)$ where $R_\odot$ is the radius of the star (in this case the sun), and a is the star–planet separation. Thus we can calculate the probability of the occultation:

$$P = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}-\theta'} sin(\theta) d\theta \tag{32}$$

$$P = -\frac{1}{2}\left[cos\left(\frac{\pi}{2}+\theta'\right) - cos\left(\frac{\pi}{2}+\theta'\right)\right] \tag{33}$$

$$P = -\frac{1}{2}\left[-sin(\theta') - sin(\theta')\right] = sin(\theta') = \frac{R_\odot}{a} \tag{34}$$

Substituting $a = 1\,\mathrm{AU}$ we get $\mathrm{P} = 8.9 \times 10^{-4}$.

c) Detecting 51 Pegasi First let us calculate the radius of 51 Peg:

$$R_* = \sqrt{\frac{L}{4\pi\sigma T^4}} = 8.3 \times 10^{10} cm = 1.2\, R_\odot \tag{35}$$

Then we assume that the orbiting planet has the same radius as jupiter, and proceed like part (a) to get,

$$\Delta f = 7.4 \times 10^{-3} = 0.74\% \tag{36}$$

Again, we have to verify that the correction due to the planets emission is small. Indeed we see that $T_J^4/T_*^4 = 1.4 \times 10^{-3}$, which is negligible.

d) Probability of finding 51 Pegasi–like systems Again proceeding like part (b), we use Equation 34 to get,

$$P_{51\,Peg} = \frac{R_*}{a_*} = 0.1 \tag{37}$$

4 Hydrogen gas

This is a straightforward application of the Boltzmann Equation. For having only 1% atoms in the first excited state,

$$10^{-2} = \frac{2 \times 2^2}{2 \times 1^2} e^{-\left[\frac{-13.6\,\mathrm{eV}}{2^2} - \frac{-13.6\,\mathrm{eV}}{1^2}\right]/kT} \tag{38}$$

Hence we get $T = 2.0 \times 10^4\,\mathrm{K}$. For having 10% atoms in the excited state, we get $T = 3.2 \times 10^4\,\mathrm{K}$.