PROBLEM SET 6

P1 : CARROLL & OSTLIE 8.9

$$n_e V = N_{II}$$

$$N_t = \frac{\rho V}{m_p}$$

$$\rho = 10^{-6}\ \text{kg m}^{-3} = 10^{-9}\ \text{g cm}^{-3}$$

a) $$\frac{N_{II}}{N_t - N_{II}} = \frac{2 \times 1}{n_e \times 2}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

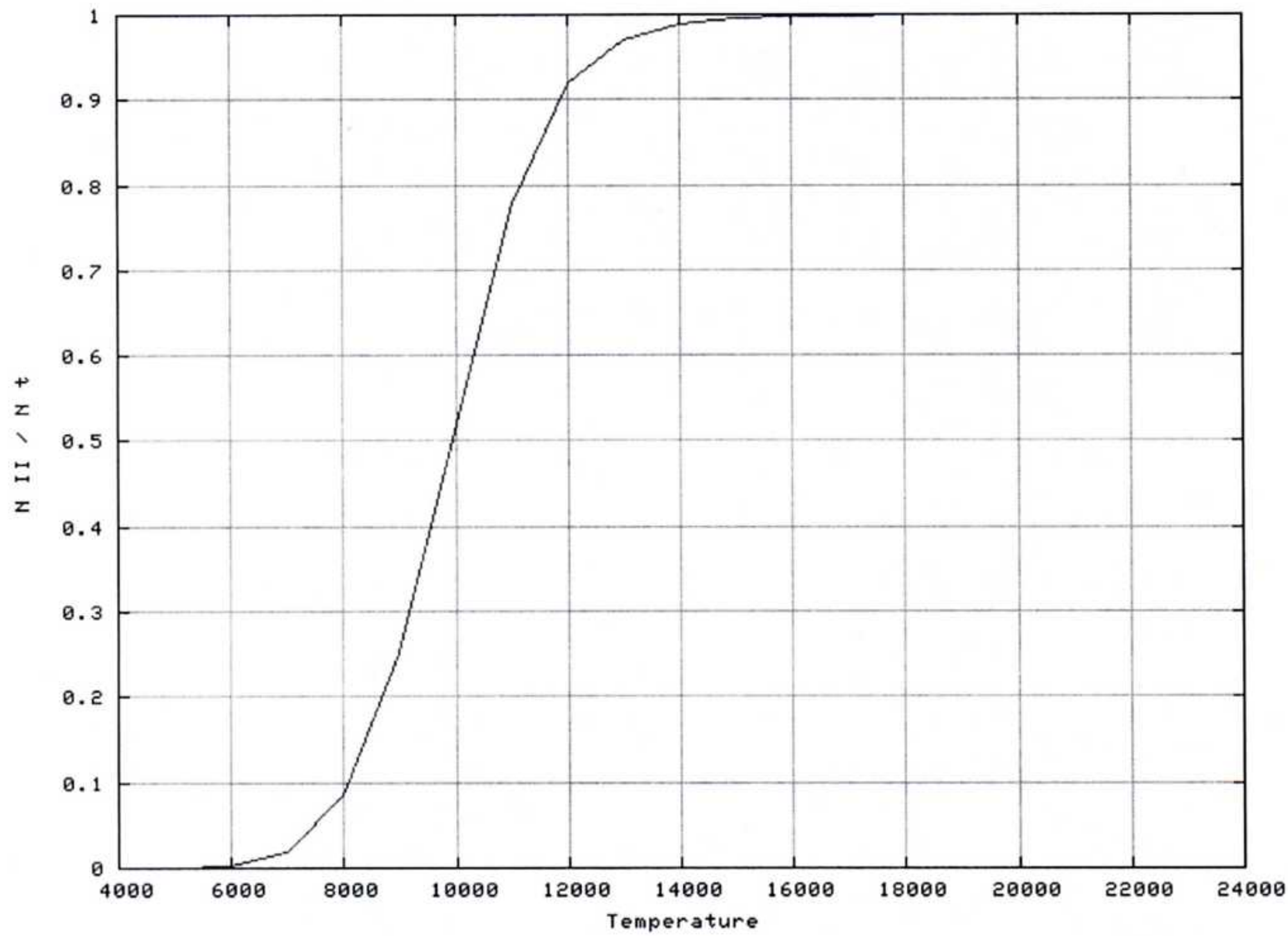
but $$n_e = \frac{N_{II}}{V} = \frac{N_{II}\,\rho}{m_p N_t}$$

$\therefore$ we get

$$1 = \left(\frac{N_t}{N_{II}} - 1\right)\frac{N_t}{N_{II}}\left(\frac{m_p}{\rho}\right)\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

$$\therefore \left(\frac{N_{II}}{N_t}\right)^2 = \left(\frac{m_p}{\rho}\right)\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

$$- \frac{N_{II}}{N_t}\left(\frac{m_p}{\rho}\right)\left(\frac{2\pi m_e kT}{h^2}\right) e^{-\chi_I/kT}$$



1) b) we have a quadratic in $x = \frac{N_{II}}{N_t}$

expressing as $ax^2 + bx + c = 0$, we get

$$a = 1$$

$$b = \frac{m_p}{\rho}\left(\frac{2\pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

$$c = -b$$

since $x = \frac{N_{II}}{N_t}$ is > 0,

$$x = \frac{-b + \sqrt{b^2 + 4b}}{2}$$

: see graph above. It matches well with 8.8

2) C&O: problem 8.10

$Z_I = 1$ $\qquad \chi_I = 24.6\ eV$

$Z_{II} = 2$ $\qquad \chi_{II} = 54.4\ eV$

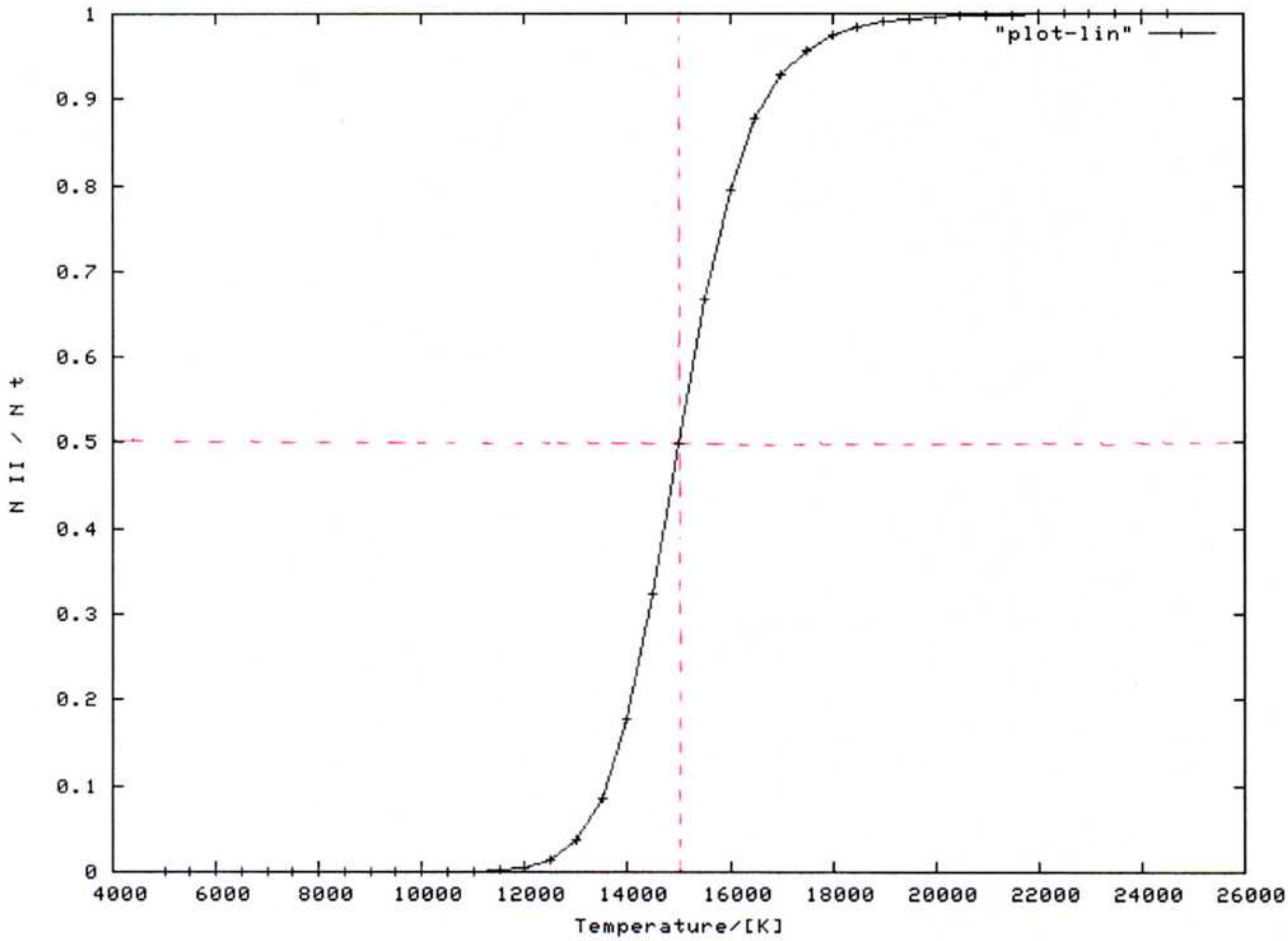
$Z_{III} = 1$

a) $\frac{N_{II}}{N_I} = ?$ $\qquad \frac{N_{III}}{N_{II}} = ?$ $\qquad$ evaluate at 5000 K, 15000 K, 25000 K

$$\frac{N_{II}}{N_I} = \frac{2kT}{P_e} \times \frac{2}{1} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

note that it is useful to substitute $k = 1.381 \times 10^{-16}$ erg K^{-1} in the expression but to use $k = 8.617 \times 10^{-5}$ eV K^{-1} in the exponential. The answers are:

Temperature	5000	15000	25000
$\frac{N_{II}}{N_I}$	1.88×10^{-18}	1.00	7.24×10^{3}
$\frac{N_{III}}{N_{II}}$	4.31×10^{-49}	2.42×10^{-11}	1.78×10^{-3}



2) b) $$\frac{N_{II}}{N_I + N_{II} + N_{III}} = \left(\frac{N_I}{N_{II}} + \frac{N_{II}}{N_{II}} + \frac{N_{III}}{N_{II}} \right)^{-1}$$

$$= \frac{1}{\left(\frac{N_{II}}{N_I}\right)^{-1} + 1 + \left(\frac{N_{III}}{N_{II}}\right)}$$

c) The graph is plotted above.
The middle of the partial ionization zone is at 15,000 K

3) C & O problem 8.14

* spectral type is based on lines seen in the star's spectrum. To get similar spectra, we need similar values of $\frac{N_{i+1}}{N_i}$

* $$\frac{N_{i+1}}{N_i} \propto \frac{T^{3/2} e^{-\chi/kT}}{n_e}$$

* it is given that giants have lower n_e
$\therefore$ numerator must be smaller in giants

* $T^{3/2}$ and $e^{-\chi/kT}$ are both MONOTONE functions of T. Hence, lower value of $T^{3/2} e^{-\chi/kT}$ implies lower T.

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4) Photons in blackbody spectrum

a) expression for $n_\lambda d\lambda$:

$$n_\lambda d\lambda = \frac{B_\lambda d\lambda}{\left(\frac{hc}{\lambda}\right)} \times \frac{4\pi}{c}$$

$$= \frac{8\pi}{\lambda^4} \cancel{d\lambda} \frac{1}{e^{hc/\lambda kT}-1} d\lambda$$

$$\therefore \int_0^\infty n_\lambda d\lambda = n = ?$$

$$\frac{hc}{\lambda kT} = x \Rightarrow \frac{-hc}{\lambda^2 kT} d\lambda = dx \Rightarrow dx = \frac{-x}{\lambda} d$$

$$\therefore n = \cancel{\frac{8\pi}{(hc)}} \; 8\pi \times \left(\frac{kT}{hc}\right)^3 \int_\infty^0 \frac{-x^2}{e^x-1} dx$$

numerically, $\int_0^\infty \frac{x^2}{e^x-1} dx = 2.404$

substituting CGS constants,

(4) $n = 20.26\, T^3$ (2) $\quad$ (n in cm^{-3}, T in K)

b) CMBR: $T = 2.726\ K \Rightarrow n = 410\ cm^{-3}$ (2)

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4) c) CMBR: $u = aT^4$

$n = 20.26\ T^3$

$\therefore \frac{u}{n} = \frac{aT}{20.26\ cm^{-3}K^{-3}}$ ~~(CGS units)~~

$a = 7.565 \times 10^{-15}$ erg $cm^{-3}\ K^{-4}$

$k = 1.381 \times 10^{-16}$ erg K^{-1}

$\therefore \frac{u}{n} = 2.704\ kT$ (2)

d) sun center : $T = 1.57 \times 10^{7}$ K

$\therefore \frac{u}{n} = 5.86 \times 10^{-9}$ ergs

$= \underline{3660\,eV}$ (1)

sun surface : $T = 5777K$

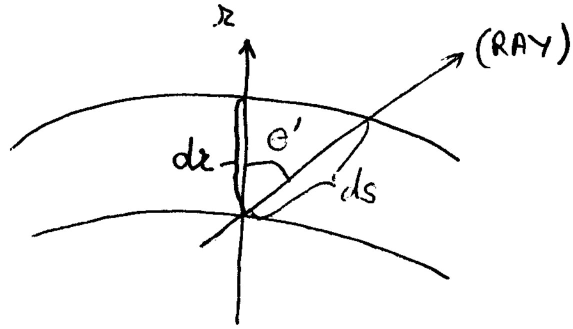
$\therefore \frac{u}{n} = 2.16 \times 10^{-12}$ ergs

~~= 1.35~~

$= \underline{1.347\ eV}$ (1)

5) C&O problem 9.16

a)



$dr = ds \cos\theta'$
from geometry of figure

but $\dfrac{dI_\lambda}{ds} \cdot \dfrac{1}{\kappa_\lambda \rho} = I_\lambda - S_\lambda$

$\therefore \quad \dfrac{-\cos\theta'}{\kappa_\lambda \cdot \rho} \dfrac{dI_\lambda}{dr} = I_\lambda - S_\lambda$

b) multiply by $\cos\theta'$ and integrate over all angles ($d\Omega$):

$$\iint_{\theta\,\phi} \frac{-dI_\lambda}{\kappa_\lambda \rho\, dr} \cos^2\theta' \, d\Omega$$

$$= \iint_{\theta\,\phi} I_\lambda \cos\theta' d\Omega - \iint_{\theta\,\phi} S_\lambda \cos\theta' d\Omega$$

moving $\frac{d}{dr}$ outside the integral assuming κ, ρ independent of r

$$-\frac{c}{\kappa_\lambda \rho} \frac{dP_{rad}}{dr} = F_{rad}$$