These problems cover Shankar 1.1-1.6, up through operators.

v. 2: Clarification: For (1c), the hint in Shankar (Exercise 1.1.1) makes it clear that you need to also show $0|v⟩ = |0⟩$. Shankar’s hint for the proof of $0|v⟩ = |0⟩$ assumes there exists a multiplicative identity element in the scalar field 1. This assumption is unnecessary. While it is certainly consistent with (1c) since the problem asks you to assume there is a 1 element, one can prove $0|v⟩ = |0⟩$ more generally. A hint on proving $0|v⟩ = |0⟩$: the definition of $|0⟩$ has nothing to do with additive inverses; it defines $|0⟩$ as the additive identity element.

Note also that our class definitions assume the uniqueness of additive identities and inverses; these do not need to be assumptions (Shankar does not assume them), but I figured there was no point in forcing you to prove these points since they are generic properties of identity and inverse elements in groups; they are not specific to vector spaces.

Sorry about the confusion; the 1st edition of Shankar, which I had in hand when I was writing the relevant notes and PS, had slightly different axioms and also did not have this misleading hint.

Clarification for (4a): You may assume the identity matrix has unit determinant.

1. Using the rules defining a linear vector space, show that

   (a) Scalar addition is commutative $α + β = β + α$ and associative $α + (β + γ) = (α + β) + γ$.

   (b) Scalar multiplication is associative, $α(βγ) = (αβ)γ$.

   (c) If the field has an element 1 that is the identity for scalar multiplication, then the 
       addition inverse of 1, denoted by $−1$, satisfies $−1|v⟩ = −|v⟩$; i.e., that $−1$ multiplying 
       a vector $|v⟩$ gives its vector addition inverse $−|v⟩$.

   See Shankar for some hints. Be careful to avoid assuming what you want to prove.

2. Consider our example vector space of real antisymmetric $3 \times 3$ matrices from the lecture notes. Let there be a set of three vectors with the first two being the $|1⟩$ and $|2⟩$ vectors from the notes. Let there be a third matrix $|C⟩$, whose elements are not yet determined. First, write down $|C⟩$ in terms of the fewest possible parameters (remember, it is real and antisymmetric). Next, write down a set of conditions on these parameters to require that $|C⟩$ be linearly independent of the first two. Then, assuming these conditions are satisfied, use Gram-Schmidt orthogonalization to obtain from $|C⟩$ a vector orthogonal to $|1⟩$ and $|2⟩$. How is the new vector related to the $|3⟩$ vector given in the lecture notes?

3. (Shankar 1.6.2): Assuming $Λ$ and $Ω$ are Hermitian operators, what can you say about

   $ΩΛ$, $ΩΛ + ΛΩ$, $[Ω, Λ]$, $i[Ω, Λ]$
4. Determinant and trace identity fun:

(a) (Shankar 1.6.4 and 1.7.2) Take it as given that the determinant of a matrix and its transpose are equal and that the determinant of a product of two matrices is the product of the individual determinants. Show that a unitary matrix has a determinant of unit modulus and that the determinant of any matrix is unaffected by a unitary transformation. (Hint: how does a determinant behave under complex conjugation?)

(b) (Shankar 1.7.1) The trace of a matrix is defined to be the sum of its diagonal elements:

\[ \text{Tr}(\Omega) = \sum_i \Omega_{ii} \]

Show the following:

\[ \text{Tr}(\Omega \Lambda) = \text{Tr}(\Lambda \Omega) \]
\[ \text{Tr}(\Omega \Lambda \Theta) = \text{Tr}(\Lambda \Theta \Omega) = \text{Tr}(\Theta \Omega \Lambda) \]
\[ \text{Tr}(U^\dagger \Omega U) = \text{Tr}(\Omega) \quad \text{where } U \text{ is unitary} \]

(Recall that matrix multiplication is equivalent to \([\Omega \Lambda]_{ij} = \sum_k \Omega_{ik} \Lambda_{kj}\).)