This problem set focuses entirely on the infinite-dimensional generalization of inner-product spaces, Shankar Section 1.10 and Lecture Notes Section 3.9.

v. 2: Clarifications/hints on Problems 1, 3, and 4 added.

1. Shankar 1.10.2 + extra: Show that

\[
\delta(f(x)) = \sum_i \delta(x - x_i) \left| \frac{df}{dx} \right|_{x=x_i}
\]

where \{x_i\} are the zeros of the function \(f(x), f(x_i) = 0\). You also should assume \(\left| \frac{df}{dx} \right|_{x=x_i} \neq 0\) and is finite for all \(x_i\) so the above formula is well-defined. Hint: where does \(\delta(f(x))\) blow up? Expand \(f(x)\) near such points in a Taylor series, keeping the first term, and do a change of integration variable. You do not need to prove the first item below in order to prove the above theorem.

Using the above, show the following:

- \(\delta(ax) = \frac{1}{|a|} \delta(x)\)
- \(\delta(x^2 - a^2) = \frac{1}{2|a|} [\delta(x-a) + \delta(x+a)]\)

(Use Equation 1 to do these even if you are unable to prove Equation 1.)

2. Use a change of integration variable to show that \(\delta(\sqrt{x}) = 0\) and use your result to evaluate \(\delta(\sqrt{x^2 - a^2})\). (This is not just a matter of applying Equation 1 again because \(\left| \frac{df}{dx} \right|_{x=0}\) is infinite.)

3. Shankar 1.10.3: Use integration by parts over an infinitesimal interval around \(x = x'\) to show

\[
\frac{d}{dx} \theta(x - x') = \delta(x - x') \quad \text{where} \quad \theta(x - x') = \begin{cases} 0 & x < x' \\ 1 & x \geq x' \end{cases}
\]

4. We have proven the following relations for derivatives of the \(\delta\) function:

\[
\left[ \frac{d}{dx} \delta(x - x') \right] = \delta(x - x') \frac{d}{dx'} \quad \left[ \frac{d}{dx'} \delta(x - x') \right] = -\delta(x - x') \frac{d}{dx'}
\]

Prove these more rigorously by using the Gaussian approximation to the \(\delta\) function and taking the appropriate limits. Note that you will have to do an integration by parts in order to make sense of the factor \(\frac{d}{dx'}\) acting to the right.
5. In class, we considered the space of complex functions with complex coefficients on the real line. Our kets $|f\rangle$ were defined via their \{|$x\rangle$\} basis representation, $\langle x | f \rangle = f(x)$. We then showed how they could be rewritten in terms of the \{|$k\rangle$\} basis by Fourier transforming:

$$\tilde{f}(k) = \langle k | f \rangle = \int_{-\infty}^{\infty} dx \langle k | x \rangle \langle x | f \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-ikx} f(x)$$

We defined the inner product $\langle f | g \rangle$ explicitly in terms of the \{|$x\rangle$\} representation:

$$\langle f | g \rangle = \int_{-\infty}^{\infty} dx \, f^\ast(x) g(x)$$

Rewrite the inner product formula in terms of the \{|$k\rangle$\} representation of $|f\rangle$ and $|g\rangle$; i.e., in terms of $\tilde{f}(k)$ and $\tilde{g}(k)$. You will need to use completeness and orthonormality.