Physics 106b – Problem Set 10 – Due Jan 28, 2005

Version 3

January 28, 2005

These problems cover the material on the dynamics of rigid bodies systems in Hand and Finch Chapter 8 and Section 5.3 of the lecture notes. Please continue to write down roughly how much time you are spending on each problem.

Changes since v. 1:

- **v. 2:** More hints on problems 3 and 4.
- **v. 3:** More hints on problem 3 in response to student questions.

1. Moment of inertia tensors (these two are almost first-year physics problems)
   (a) Hand and Finch 8.9. Note: do this in the frame that the masses' positions are given in. Do **not** do this in the center-of-mass frame. Though you may find it interesting to compare the principal axes you get to what they would be relative to the center of mass – you can guess the principal axes relative to the center of mass by symmetry. This is an interesting example of how the principle axes depend on the origin you choose.
   (b) Hand and Finch 8.11. All three masses are pointlike. You don't really need to use the displaced axis theorem (you can do this problem much like Hand and Finch 8.9), but you might want to want to use it just to get some practice. Note that, for this problem, you **do** want to calculate the inertia tensor relative to the center of mass.

2. Hand and Finch 8.12. Note that there are two ways to do parts b and c: i) separate the translational motion of the center of mass from the rotational motion about the center of mass and calculate the kinetic and potential energy of each part separately; and ii) treat the cylinder as instantaneously rotating about the line of contact with the plane, which lets you calculate the kinetic energy as purely rotational and the effect of gravity as a pure torque (which can be integrated to obtain a potential energy). You may use whichever method you prefer, but it's a good idea to learn how to do it the latter way so you get used to that way of thinking.

3. Hand and Finch 8.18. Do not obtain full solutions for the time dependence of all the angles after the perturbation; it gets pretty gory. Just find the new equilibrium position, demonstrate that the system is stable, and obtain the time evolution of the deviation $\delta\theta(t)$ from the new equilibrium given the initial impulses $\delta\dot{\theta}_0$ and $\delta\dot{\phi}_0$. Hints:
   (a) A Lagrangian approach, like at the start of Section 8.10, is a convenient one for finding the conserved quantities and equations of motion.
   (b) Note that $p_\phi = \mathrm{I}_1\,\omega_P = L$ and $p_\psi = \mathrm{I}_3\,\omega_3 = p_\phi\,\cos\theta$ are conserved quantities in the symmetric top problem (except possibly at the instant that the impulses are applied).

(c) On how to deal with the impulses – think about the way we dealt with such impulsive changes in velocity in Problem Set 8, problem 1, and when dealing with Green's functions for the SHO last term. Some quantities may undergo step changes, others must be continuous.

(d) Since the equilibrium position changes, don't try to demonstrate stability about the old equilibrium – you will get very frustrated!

Answers to student questions:

- **Q:** Where do you get the relationship $p_\psi = p_\phi \cos\theta$? Looking in H&F p314 Eq 8.96, there appears to be a constant term out front ($I_1 \sin^2\theta$) that isn't included in your identity. Is there a small angle approximation going on here ($\sin\theta \approx 0$)?
  A: That relationship holds when there is no torque – see Section 5.3.3 of the notes, "Conserved Quantities" – when you can assume $\ddot{\theta} = 0$. It does not in general hold for the top with torque, or for the top without torque if not at equilibrium, but it will hold for the torque-free top at any equilibrium position.
- **Q:** We talked to the TA and after working on it, we arrived at the point where $\delta\theta_0$ is non-zero. The problem is how can there be an instantaneous change in position after impulse.
  A: That's ok if the equilibrium position changes – then the absolute position (θ) can be continuous yet $\delta\theta$ changes from 0 to a nonzero value because the equilibrium position has shifted.

More hints (really, a guide to doing the problem) (v. 3):

- First, forget about this problem, just think about a torque-free top. In the torque-free problem, we know from Section 5.3.2 of the lecture notes that there is an equilibrium value of the polar angle – one that, if you set the initial conditions there, the system stays there with no motion. This is clearly a value of θ for which $\ddot{\theta}$ vanishes. This is a (stable or unstable) equilibrium position. Section 5.3.3 obtains a relationship between the canonical momenta p_ϕ and p_ψ and the equilibrium position: $p_\psi = p_\phi \cos\theta_0$ if θ_0 is the equilibrium position.
- Suppose the equilibrium position is θ_0. An equilibrium position may be stable or unstable. How does the equation of motion behave for small deviations from θ_0? Does the equation of motion imply stability or instability against small perturbations in θ? As a hint, recall that we derived the equation of motion for θ for the symmetric top under a gravitational torque in Section 5.3.3, "Effective Potential and 1D Equation of Motion." If we just drop the gravitational torque from that equation of motion, we have:

$$\mathrm{I}_{1d}\,\ddot{\theta} = -\frac{\partial}{\partial\theta}\left(\frac{1}{2\,\mathrm{I}_{1d}}\frac{(p_\phi - p_\psi\cos\theta)^2}{\sin^2\theta} + \frac{1}{2\,\mathrm{I}_3}\,p_\psi^2\right)$$

- Now, returning to the problem. For $t < 0$, we have a torque-free top. Thus the system is in its equilibrium state – it has some fixed polar angle θ_0, its polar angle velocity $\dot{\theta}$ vanishes, it has some fixed values of the canonical momenta p_ϕ and p_ψ. It precesses at some speed $\dot{\phi}$, and it spins at some speed $\dot{\psi}$. Then we have the impulses $\delta\dot{\theta}_0$ and $\delta\dot{\phi}_0$ at $t = 0$. As we have seen both with the SHO and in problem set 8, problem 1, some quantities change discontinuously but other quantities must be continuous when such an impulse is applied. The full list of physical variables that parameterize the system are: θ, $\dot{\theta}$, ϕ, $\dot{\phi}$, ψ, $\dot{\psi}$. Which of these change discontinuously at $t = 0$, which remain continuous? What about the canonical momenta p_ϕ and p_ψ (remember, they are defined in terms of the aforementioned 6 variables)?

- Given that some quantities change at $t = 0$, does the equilibrium position in θ change? Remember that, for $t > 0$, the top is again torque-free, so everything we derived about the equilibrium behavior of the top in the absence of torque again holds.
- Now, combine all the above information. We have the pre-impulse equilibrium position θ_0. We have a possibly different post-impulse equilibrium position θ_*. We have the values of all the variables θ, $\dot{\theta}$, ϕ, $\dot{\phi}$, ψ, and $\dot{\psi}$ **just after the impulse** in terms of their pre-impulse values and the impulses. We have the equation of motion for θ in the absence of torques. That equation of motion depends on more parameters than just θ, but above you have figured out how the impulses change those parameters. So, there is now enough information – an equation of motion and initial conditions – to obtain $\delta\theta(t) = \theta(t) - \theta_*$.

4. (Symon 11.15) A top consists of a disk of mass M, radius r, mounted at the center of a cylindrical axle of length l, radius a, where $a \ll l$, and negligible mass. The end of the axle rests on a table, as shown in the figure below. The coefficient of sliding friction is μ. The top is set spinning about its symmetry axis with a very great angular velocity $\omega_{3,0}$ and is released with its axis at an angle θ_1 with the vertical. Assume that ω_3 is great enough compared with all other motions of the top so that the edge of the axle in contact with the table slides on the table in a direction perpendicular to the top axis, with the sense determined by ω_3. Assume that the nutation is small enough to be neglected, and that the friction is not too great, so that the top precesses slowly at polar angle θ_0, which changes slowly due to the friction with the table. Show that the top axis will rise to a vertical position, and that the approximate time required is $t = r^2\,\omega_{3,0}\,\theta_1/\mu\,g\,l$.

   Note that Hand and Finch 8.20 is a variant on this problem. Don't let it throw you off track, though – there's more going on in that problem, it's much harder.