Physics 106b – Midterm Exam – Due Feb 10, 2006

Instructions

Material: All lectures through Jan 31, lecture notes Section 4, 5, and 6, excluding Sections 5.1.5 and 6.1.3. You are of course also responsible for the fundamental material from Ph106a that was used in the material we covered this term (Lagrangian/Hamiltonian mechanics, simple oscillations, etc.). Review the material ahead of time, consult me, the TAs, your fellow students, or other texts if there is material you are having trouble with.

Logistics: The exam consists of this page plus 4 pages of exam questions. Do not look at the exam until you are ready to start it. Please use a blue book if possible (makes grading easier), but there will be no penalty if you don't have one. Problems 1 through 6 are required, Problem 7 is extra credit.

Time: 4 hrs, fixed time. You may take as many breaks as you like, but they may add up to no more than 30 minutes (2 x 15 minutes, 3 x 10 minutes, etc.).

Reference policy: Hand and Finch, Thornton (including solutions manual), official class lecture notes and errata, problem sets and solutions, your own lecture notes or other notes you have taken to help yourself understand the material. No other textbooks, no web searches, no interaction with your fellow students. Calculators and symbolic manipulation programs are neither needed nor allowed.

Due date: Friday, Feb 10, 5 pm, my office (311 Downs). 5 pm means 5 pm. Late exams will require extenuating circumstances; otherwise, no credit will be given.

Grading: The point value for each problem is listed by subpart. The total for the required problems is 100 points. The exam is 1/3 of the class grade.

Semantics: If the problem says something like "Show that the expression for such-and-such is" and then gives you the result, you are responsible for deriving that result as if you did not know it ahead of time, not just for plugging it in to see that it is indeed correct. The results are given so you can check whether you have done the problem incorrectly and also to make it possible to do the subparts of a problem out of order.

Suggestions on taking the exam:

- Don't let the length or the number of problems bother you: (1) and (2) are short and, as you know, I tend to err on the side of verbosity.
- Go through and figure out roughly how to do each problem first; make sure you've got the physical concept straight before you start writing down formulae.
- Don't fixate on a particular problem. Finish the ones you find straightforward first, then come back to ones you are having difficulty with.
- The problems are split up into subparts that can usually be done out of order. If you can't get one part, move on to the next one and come back later.
- Don't get buried in algebra. Get each problem to the point where you think you will get most of the points, then come back and worry about the algebra.

1. (10 pts) Show that the expression for the Lorentz contraction can be obtained by the following alternate method. Let F' be the rest frame of a rod of length L'. Let F' be moving at speed v in the $+\hat{x}$ direction relative to another frame F, and let their origins coincide at $t = t' = 0$. Let the observer in F make a length measurement of the rod by measuring the time between two events at the F origin – namely, the passing of the front end of the rod and the back end of the rod – and estimating the rod length by multiplying this time by the speed v at which the rod is moving. You may set $c = 1$ if you like.

2. (10 pts) The Bevatron at Berkeley was built with the idea of producing antiprotons by the reaction $p+p \longrightarrow p+p+p+\overline{p}$. That is, a high-energy proton strikes a proton at rest, creating (in addition to the original particles) a proton-antiproton pair. Let p_A^μ and p_B^μ indicate the energy-momentum four-vectors of the incident proton and the at-rest proton. By considering the problem in both the lab frame (in which proton B is at rest) and the center-of-momentum frame (the frame in which the spatial component of the total energy-momentum four-vector $p_A^\mu + p_B^\mu$ vanishes), find the minimum center-of-momentum-frame energy E_A^{CM} and minimum lab-frame energy E_A^{lab} that the incoming proton must have in order for the reaction to proceed. Your answer should be in terms of the proton mass m_p (and possibly factors of the speed of light c). You may set $c = 1$ if you prefer. Note that the mass of the antiproton is equal to that of the proton. You will likely find it useful to know that

$$\left(p_A^{\,\mu} + p_B^{\,\mu}\right)\left(p_{A\,\mu} + p_{B\,\mu}\right) = (E_A + E_B)^2 - |\gamma_A\, m_A\, \vec{v}_A + \gamma_B\, m_B\, \vec{v}_B|^2$$

is invariant under Lorentz transformations, a fact that is true simply because the quantity is the invariant norm of the four-vector $p_A^\mu + p_B^\mu$.

3. Consider a repulsive central force inversely proportional to the cube of the radius,

$$F(r) = \frac{K}{r^3} \qquad\qquad K > 0$$

   (a) (10 pts) Show that the orbits are of the form

$$\frac{1}{r} = A\cos\alpha\theta \qquad \text{with} \quad \alpha = \sqrt{1 - \frac{\mu\, K}{l^2}}$$

   (b) (10 pts) Show that the differential cross section is

$$\frac{d\sigma}{d\Omega}(\theta_*) = \frac{1}{\sin\theta_*}\,\frac{\pi^2 K}{m\, v_\infty^2}\,\frac{\pi - \theta_*}{\theta_*^2\,(2\,\pi - \theta_*)^2}$$

   where v_∞ is the incoming beam velocity (particle velocity as $r \to \infty$) and θ_* is the scattering angle as defined in the lecture notes. Realize that you do not need to solve for the full orbit shape or do a horrible integral to obtain the function $b(\theta_*)$ needed to calculate the cross section.

4. A rocket is in an elliptical orbit around the earth, with distances of closest and furthest approach r_1 and r_2 (also known as the radial turning points of the motion, or perigee and apogee). (Assume the earth and rocket are point-like.) At a certain point in its orbit, the rocket's engine is fired for a short time so as to give a velocity increment Δv in order to put the rocket on an orbit that escapes from the earth with a final velocity v_∞ relative to the earth.

(a) (10 pts) Show that Δv is a minimum if the thrust is applied at perigee (r_1), parallel to the orbital velocity.

(b) (10 pts, and lots of opportunity for partial credit) Find Δv in that case in terms of the orbit eccentricity ϵ, the orbit semimajor axis a, Newton's constant G, the reduced mass μ and total mass M of the rocket-earth system, and the final velocity v_∞ (*i.e.*, your final result should not use r_1 or r_2). Can you explain physically why Δv is smaller for larger ϵ?

5. Referring back to Section 7.8 of Hand and Finch regarding the deflection of a falling point particle *relative to a plumb line*, we had that the Coriolis-induced velocity and deflection of the particle to first order in ω were

$$\Delta\vec{v}^{(1)} = \omega\, g\, t^2\, \cos\lambda\,\hat{x}$$
$$\Delta\vec{r}^{(1)} = \omega\, g\, \frac{t^3}{3}\, \cos\lambda\,\hat{x}$$

where t is the time since the particle was released. Take g to be defined at ground level and use the zeroth-order approximation for the time-of-fall, $T = \sqrt{2\,h/g}$, where h is the height from which the particle is dropped. Calculate the southerly deflection due to two effects that are of second order in ω:

(a) Coriolis force to second order (C_1)

(b) Variation of centrifugal force with height (C_2)

Show that each of these components gives a result equal to

$$\Delta\vec{r}^{(2)} = -C_i\frac{h^2}{g}\,\omega^2\sin\lambda\,\cos\lambda\,\hat{y}$$

(assuming we define $+x$ to be east and $+y$ to be north as in Hand and Finch) with $C_1 = 2/3$ and $C_2 = 5/6$. Each term is worth 5 pts.

6. Consider a charged sphere whose mass m and charge q are both distributed in a spherically symmetric way. That is, the mass and charge densities are each functions of the radius r (but not necessarily the same function). Do the following:

(a) (8 pts) Show that, if the body rotates in a uniform magnetic field $\vec{B}$, then the torque on it is

$$\vec{N} = \frac{q\,g}{2\,m\,c}\,\vec{L}\times\vec{B}$$

in Gaussian units, where g is a numerical constant of order unity (called the gyromagnetic ratio). You will find it useful to take a look at the vector algebraic identities in Section A.3 of the lecture notes. You will also likely run into an expression of the form

$$\int d^3r\, f(r)\,(\vec{a}\times\vec{r})\left(\vec{r}\cdot\vec{b}\right)$$

where $\vec{a}$ and $\vec{b}$ do not depend on $\vec{r}$. Realize that this can be written as

$$\vec{a}\times\left[\int d^3r\, f(r)\,\vec{r}\,\vec{r}^{\,T}\right]\vec{b}$$

The expression in brackets is similar in form to something we have seen this term.

(b) (2 pts) Show $g = 1$ if the mass density is everywhere proportional to the charge density.

(c) (5 pts) Write an equation of motion for the angular momentum of the body and show that, by introducing a suitable rotating coordinate system, you can eliminate the magnetic torque.

(d) (3 pts) Refer back to Larmor's theorem (PS10, #2), where we had to place a "weakness" condition on $\vec{B}$ to obtain a similar result. Why is no such condition needed here?

(e) (5 pts) Describe the motion – *i.e.*, how do $\vec{L}$ and $\vec{\omega}$ behave? A new angular velocity should arise – compute its value. Why is it not necessary to take into account this new angular velocity when computing the magnetic torque?

(f) (7 pts) An electron may for some purposes be regarded as a spinning charged sphere of the kind considered in this problem, with $g \approx 2$. Show that, if g were exactly equal to 2, and the electron's spin angular momentum is initially parallel to its linear velocity, then its spin angular momentum would remain parallel to its velocity as it moves through *any* (*i.e.*, possibly nonuniform) magnetic field (assuming the magnetic field is uniform on the size scale of the electron itself). Deviations from $g = 2$ are the result of "virtual particle" effects, such as the electron emitting and reabsorbing a virtual photon (or other, more massive particles such as the carriers of the weak force). A recent experiment measured $g - 2$ of the muon using the behavior you discovered in part (e) and the fact that the decay of the muon to an electron, muon neutrino, and electron antineutrino is correlated with the direction of the muon spin. They found $g - 2$ to be somewhat out of agreement with Standard Model predictions, suggesting the existence of new particles outside the Standard Model.

7. **(Extra credit)** A more detailed investigation of the nutating solutions for a symmetric top with torque.

We have seen that the θ turning points of the nutating solution to the heavy symmetric top problem are given by the roots of a third-order polynomial obtained from the expression for the energy of the top. Suppose we give the top the particular initial condition,

$$\theta = \theta_1 \qquad \dot{\theta} = 0 \qquad \dot{\phi} = 0 \qquad \dot{\psi} = \omega_3$$

where θ_1 and ω_3 are constants set by the user. This is a pretty typical initial condition, consisting of setting the top spinning at angular speed ω_3, tipping at some angle θ_1 from vertical, and then releasing. In this case, θ_1 is one of the roots of the polynomial. Do the following:

(a) (3 pts) Use this fact to show that the other two roots of the polynomial are

$$\cos\theta = \frac{1}{2\,\alpha}\left[1 \pm \sqrt{1 - 4\,\alpha\,\cos\theta_1 + 4\,\alpha^2}\right] \qquad\qquad \alpha \equiv \frac{2\,I_{1d}\,M\,g\,l}{I_3^2\omega_3^2}$$

Why must the root with the plus sign be discarded?

(b) (2 pts) Assume the top is spinning very rapidly, so that $\alpha \ll 1$. Show that the physical root then becomes

$$\cos\theta_2 = \cos\theta_1 - \alpha\,\sin^2\theta_1$$

(c) (5 pts) For $\alpha \ll 1$, the amplitude of the motion in θ is very small. Treating the motion as simple harmonic oscillation about $\theta_0 = \frac{1}{2}(\theta_1 + \theta_2)$, show that the amplitude of the motion is $a = \frac{1}{2}\alpha \sin\theta_1$. Taylor expand the effective potential to second order in the deviation $\delta = \theta - \theta_0$ to show that the frequency of the oscillation in δ is

$$\omega_\theta = \frac{I_3}{I_{1d}}\,\omega_3$$

(d) (5 pts) Given the above oscillation in θ, obtain the mean value and amplitude of the resulting oscillation in $\dot{\phi}$. Sketch the shape of the resulting nutation (*i.e.*, the path of the 3-axis of the top) in a manner similar to Figure 8.17 of Hand and Finch.