Physics 106a/196a – Problem Set 4 – Due Oct 28, 2005

These problems cover the material on symmetry transformations in Section 2.1.9 of the lecture notes and on the calculus of variations and Hamilton's Principle in Sections 2.2.1 and 2.2.2 of the lecture notes. Some of the problems are general practice on Lagrangians, reaching back to Sections 2.1.1 to 2.1.8 of the notes. Problems 1 and 2 are for 106a students only, 3 and 4 for 106a and 196a students, and 5 and 6 for 196a students only.

1. (106) Hand and Finch 1-19, but do not try to obtain full solutions, just find the Euler-Lagrange equations of motion for the two cases. Note that, for the second case, the Euler-Lagrange equation for the translational motion provides a simple relation between the translation and the rotation, but, even after substituting this into the rotational Euler-Lagrange equation, the latter differential equation is nevertheless not trivially integrable.

2. (106) A particle of mass m rests on a smooth plane. The plane is raised to an inclination angle θ at a constant rate α ($\theta = 0$ at $t = 0$), causing the particle to move down the plane. Define an appropriate set of generalized coordinates and obtain the Euler-Lagrange equation of motion for the system. (Optional: if you want to practice solving inhomogeneous differential equations, give this one a shot; it's kind of amusing. The result will be provided in the solutions.)

3. (106/196) The last Noether conserved quantity for simple systems. Suppose one has a system of particles, possibly interacting with each other, but subject to no external potentials or forces. The Lagrangian can be written in terms of center-of-mass motion and coordinates relative to the center of mass:

$$L\left(\vec{R}, \{\vec{s}_a\}, \dot{\vec{R}}, \{\dot{\vec{s}}_a\}\right) = \frac{1}{2}\, M\, \dot{\vec{R}} \cdot \dot{\vec{R}} + \sum_a \frac{1}{2}\, m_a\, \dot{\vec{s}}_a \cdot \dot{\vec{s}}_a - \sum_{b>a} U_{ab}\left(|\vec{s}_b - \vec{s}_a|\right)$$

$$\text{with} \qquad \vec{R} = \frac{1}{M}\sum_a m_a \vec{r}_a \qquad \vec{s}_a = \vec{r}_a - \vec{R} \qquad M = \sum_a m_a$$

Consider transformations of the form $\vec{r} \to \vec{r}\,'$ with $\vec{r}\,' = \vec{r} - \vec{v}\,t$ where $\vec{v}$ is some arbitrary constant velocity. This is a transformation into an inertial reference frame F' moving with velocity $\vec{v}$ with respect to the initial inertial reference frame F (also known as a "boost"; we will consider these in great detail when we come to special relativity).

   (a) Instead of using the $\{\vec{r}_a\}$ as our coordinates, we have changed to $\vec{R}$ and $\vec{s}_a$ in the above L. Find the Euler-Lagrange equation of motion in the $\vec{R}$ coordinate. (Things are slightly more complex for the $\{\vec{s}_a\}$ because they satisfy the constraint $\sum_a m_a \vec{s}_a = 0$. But we will see we won't need to worry about the $\{\vec{s}_a\}$.)

   (b) What is the transformation rule for the $\vec{R}$ and $\{\vec{s}_a\}$ under the boost? That is, if $\vec{R}\,'$ and $\{\vec{s}\,'_a\}$ denote the center-of-mass coordinate and the coordinates relative to the center of mass in F', how are they related to the corresponding coordinates in F?

(c) Find how the Lagrangian changes under the transformation: obtain $L'\left(\vec{R}', \{\vec{s}'_a\}, \dot{\vec{R}}', \{\dot{\vec{s}}'_a\}\right)$ from the original Lagrangian. You will find that the transformation is not an exact symmetry transformation – that is, $L'\left(\vec{R}', \{\vec{s}'_a\}, \dot{\vec{R}}', \{\dot{\vec{s}}'_a\}\right) \neq L\left(\vec{R}', \{\vec{s}'_a\}, \dot{\vec{R}}', \{\dot{\vec{s}}'_a\}\right)$ – because extra terms appear involving the center-of-mass velocity. Show that the Euler-Lagrange equations for the center-of-mass coordinate are, nevertheless, unchanged.

(d) Even the though the transformation is not a perfect symmetry transformation, it is pretty close. Use the results derived in class (without consideration for the fact that the symmetry is imperfect) to obtain the Noether "pseudo-conserved quantities" for the transformation. Note that there are three parameters of the transformation, the three components of $\vec{v}$. The pseudo-conserved quantities are time-dependent, which occurs because our transformation was not a perfect symmetry transformation and dependent on time. Can you give a physical interpretation of these pseudo-conserved quantities? In what sense are they pseudo-conserved?

(e) Go back to the derivation of Noether's theorem in the notes and explain how the derivation would be affected by the fact that the symmetry transformation is imperfect and what kind of additional terms appear, and why these additional terms are consistent with having the above time-dependent pseudo-conserved quantities.

4. (106/196) Hand and Finch 2.7. This type of situation also occurs from ionospheric reflection of AM radio waves (you will learn about the density-dependent index of refraction of an ionized plasma in Ph106c.)

5. (196) Consider a single-particle system subject to a velocity-dependent potential as discussed in class. Let the coordinate dependence of the potential be such that it is invariant under rotations about a direction $\hat{n}$, so that the Lagrangian is invariant under such rotations. In the absence of velocity-dependent terms, the Noether conserved quantity would simply be the mechanical angular momentum about $\hat{n}$, p_θ. Show that the Noether conserved quantity when the velocity-dependent terms are included is

$$\pi_\theta = p_\theta - \hat{n} \cdot \left(\vec{r} \times \vec{\nabla}_{\vec{v}}\, U\right)$$

where $\vec{\nabla}_{\vec{v}}$ is the gradient of U with respect to velocities, rather than coordinates, as defined in the lecture notes. Notice how p_θ comes from $\frac{\partial T}{\partial \dot{\theta}}$ and the additional term comes from $\frac{\partial U}{\partial \dot{\theta}}$. Hint: how does an infinitesimal rotation $\delta\dot{\theta}$ about $\hat{n}$ affect the velocity of the particle?

For the electromagnetic case, the resulting conserved momentum would be

$$\pi_\theta = p_\theta + \hat{n} \cdot \left(\vec{r} \times \frac{q}{c}\vec{A}\right)$$

where q is the charge of the particle and $\vec{A}$ is the vector potential.

6. (196) Hand and Finch 2.6 (Laplace's Equation via variational principle). In part (a), we note that the condition $\delta\Phi = 0$ on the boundary is entirely realistic because this is how such problems in electrostatics are usually specified – a potential is set up on the boundary, we want to know the potential inside the region. In part (b), it is stated "Do this in cartesian coordinates." This does not imply you should write out all the derivatives in gory detail in cartesian coordinates; it is only intended to make the manipulation of the gradients simple.