Physics 106a/196a – Problem Set 6 – Due Nov 18, 2005

Version 2 – Nov 13, 2005

We have a bit of a hodgepodge this week. Because of the two 196a-material lectures, there is not much new 106a material. So we can use this set to catch up bit and also give you a refresher on the SHO. We cover Liouville's theorem and some basic SHO problems. We also begin exploring the damped SHO in connection with the concept of adiabatic invariance. For 196a, we skip Liouville's theorem and the basic SHO in favor of some material on Poisson brackets and the Hamilton-Jacobi equation.

Problems 1 and 2 are for 106a students only, 3 and 4 for 106a and 196a students, and 5 and 6 for 196a students only.

v. 2 contains additional hints.

1. (106) Liouville's theorem and optics. An optical system can be thought of as a system that manipulates a beam of photons. At any optical element, the beam of photons has some cross-sectional area A and the photons' momenta (wavevectors) fill some solid angle Ω. Liouville's theorem tells us that $A\,\Omega$ is conserved. You are also well aware from Ph12-level diffraction theory that a telescope of diameter D focuses photons from a distant source down to a spot of effective diameter $f\lambda/D$ on its focal plane, where f is the focal length of the telescope. Use the latter fact in the case of a simple telescope consisting of a paraboloidal mirror to determine what value $A\,\Omega$ takes on, up to a constant of order unity. (If you're careful, you can actually show that the constant is 1.). The fact that $A\,\Omega$ has a particular value is known as the **throughput theorem**; the quantity $A\,\Omega$ is known as the **optical throughput**, or, if you are feeling fancy, the **étendue**. Hint on obtaining the correct constant factor: The $\vec{E}$-field pattern in the focal plane is just that due to diffraction by a circular hole,

$$|\vec{E}(r)| = 2\,\frac{J_1(\pi\,r_0)}{\pi\,r_0}$$

   where $r_0 = f\,\lambda/D$ in our case and J_1 is a Bessel function. A is obtained on the focal plane by integrating the power, $P = |\vec{E}|^2$, over the focal plane. You can find this integral of the Bessel function in tables or using Mathematica.

2. (106) Hand and Finch 3-5.

3. (106/196) Hand and Finch 3-10. A couple hints:

   - Be careful with the initial conditions. The position cannot change discontinously, but the velocity may if the force is impulsive. Application of the initial condition is very similar to deriving the Green's functions for the various damped SHO cases.
   - When comparing the underdamped and overdamped solutions to the critically damped solution, you will have to take limits as $Q \to \frac{1}{2}$. Be sure to do these carefully, taking into account all Q dependences and Taylor expanding where necessary.

4. (106/196) Hand and Finch 3-12.

5. (196) Practice with Poisson brackets; relation of Poisson brackets to symmetry transformations.

   (a) Hand and Finch 6-15

   (b) Hand and Finch 6-16

   You will find it much more convenient to do these problems using the Einstein summation convention, the Kronecker symbol δ_{ij}, and the completely antisymmetric Levi-Civita symbol ϵ_{ijk}. In particular, it would let you see using a single derivation that $[\phi, l_i] = 0$ for all three values $i = x$, y, and z. See Appendices A.1 and A.3 of the lecture notes. Note that Equation A.20 has a typo, it should be

$$\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}$$

6. (196) In the version of Hamilton-Jacobi theory developed in class and in the textbook, a generating function of the type $G_2(q, P, t)$ is sought that generates a canonical transformation to anew canonical variables Q and P, which are constant of the motion. But the coice of this type of generating function was arbitrary. In this problem, you are to pursue the alternative of finding a generating function of the type $G_3(Q, p, t)$.

   (a) Consider a system with one degree of freedom and Hamiltonian $H(q, p)$. Write down a Hamilton-Jacobi partial differential equation for a function $T(p, t)$ such that the solution $T(Q, p, t)$ (depending on the one constant of integration Q) generates a canonical transformation from q and p to new variables Q and P that are constant.

   (b) For a particle moving vertically in a uniform gravitational field, the Hamiltonian is

$$H = \frac{p^2}{2\,m} + m\,g\,q$$

   What is the Hamilton-Jacobi equation for $T(p, t)$ in this case?

   (c) Finda solution to this equation of the form

$$T(p, t) = W(p) - Q\,t$$

   where Q is a constant.

   (d) Find the canonical transformation generated by $T(Q, p, t)$. That is, express Q and P in terms of q, p, and t, and invert these expressions to find q and p in terms of Q, P, and t.