Physics 106a/196a – Final Exam – Due Dec 8, 2006

Instructions

Material: All lectures through Nov 28; see syllabus for details. Only Ph196 students are responsible for the following topics:

- virtual work and generalized forces
- derivation of Euler-Lagrange equations via virtual work
- nonholonomic constraints
- Lagrangians for nonconservative forces
- incorporating nonholonomic constraints via Lagrange multipliers
- canonical transformations, generating functions, symplectic notation, Poisson brackets, Hamilton-Jacobi theory
- mathematical structure of coupled oscillations

Ph106a students **are** responsible for action-angle variables and adiabatic invariance, but only the use thereof – not the derivations.

Review the material ahead of time, consult me, the TAs, your fellow students, or other texts if there is material you are having trouble with.

Logistics: The exam consists of this page plus 3 pages of exam questions. Do not look at the exam until you are ready to start it. Please use a blue book if possible (makes grading easier), but there will be no penalty if you don't have one. Problems 1 and 2 are for 106 students only, 3, 4, and 5 for 106 and 196 students, and 6 and 7 for 196 students only.

Time: 4 hrs, fixed time. You may take as many breaks as you like, but they may add up to no more than 30 minutes (2 x 15 minutes, 3 x 10 minutes, etc.).

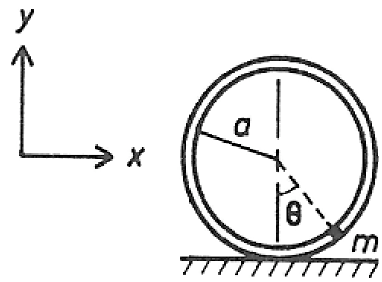
Reference policy: Hand and Finch, official class lecture notes and errata, problem sets and solutions, your own lecture notes or other notes you have taken to help yourself understand the material. No other textbooks, no web searches, no interaction with your fellow students. Calculators and symbolic manipulation programs are neither needed or allowed.

Due date: Friday, Dec 8, 5 pm, my office (311 Downs). 5 pm means 5 pm. Late exams will require extenuating circumstances; otherwise, no credit will be given.

Grading: Each problem is 20 points out of 100. The exam is 1/3 of the class grade.

Suggestions on taking the exam:

- Go through and figure out roughly how to do each problem first; make sure you've got the physical concept straight before you start writing down formulae.
- Don't fixate on a particular problem. They are not all of equal difficulty. Come back to ones you are having difficulty with.
- Don't get buried in algebra (this really should not be an issue on this exam). Get each problem to the point where you think you will get most of the points, then come back and worry about the algebra.



Problem 1

1. (106) A circular track of radius a and mass M sits on a frictionless table as shown in the figure. A particle of mass m is confined to move without friction on the circular track. All motion is confined to a vertical plane. Note that there is no friction, so the track does not roll, it can only slide.

   (a) (7 pts) Set up the generalized coordinates and constraint equations in such a way as to allow you to find via Lagrange multipliers for the radial constraint force keeping m on the circular track and the vertical constraint force keeping the track on the table.

   (b) (7 pts) Write down the Lagrangian and find the equations of motion (no small-angle approximation). Be sensible about when to explicitly take the time derivatives of the canonical momenta and when to leave them unexpanded as $\frac{d}{dt}(p_k)$.

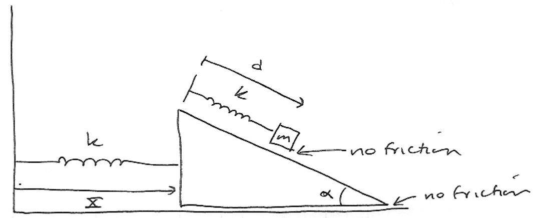
   (c) (6 pts) For small angular displacements about $\theta = 0$, find the accelerations, the Lagrange multipliers, and the constraint forces. Eliminate $\dot{\theta}$ from all expressions using conservation of energy, and eliminate $\dot{x}$ from as many equations of motion as possible. You may assume the system begins at rest.

2. (106) Grab bag of quickie problems. These should require a quite minimal amount of calculation, don't let yourself head off in the wrong direction and get bogged down!

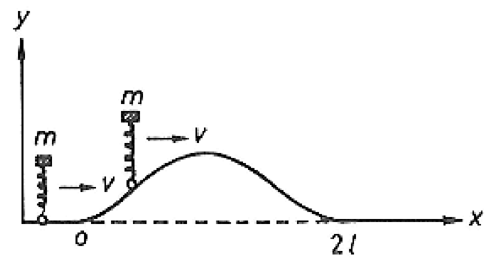
   (a) (10 pts) An hourglass sits on a weighing scale (like a bathroom scale). Initially, all the sand (mass m) in the hourglass (mass M) is in the upper reservoir. At $t = 0$, the sand is released and exits the upper reservoir at constant rate $\dot{m}_{upper} = -\lambda$, $\lambda > 0$. You may assume all grains of sand fall the same distance h from the bottom of the top reservoir to the bottom of the bottom reservoir (*i.e.*, no sandpile builds up). Draw a qualitative graph showing the reading of the scale for all times $t > 0$, with quantitiative labels on the time and force axes (in terms of m, M, λ, g, and h).

   (b) (5 pts) A particle of mass m, charge q, and initial velocity v collides head-on with an identical particle initially at rest, where both velocities are given in the lab frame. What is the distance of closest approach between the two particles? What are the velocities of the two particles relative to the lab frame at this moment of closest approach?

   (c) (5 pts) Two stars with masses M and m separated by a distance d are in circular orbits around the stationary center of mass. The stars may be treated as point masses. In a supernova explosion, the star of mass M loses a mass ΔM. The explosion is instantaneous, spherically symmetric, exerts no reaction force on the remnant star left behind, and has no direct effect on the other star. Show that the binary system remains bound only if $\Delta M < \frac{1}{2}(M + m)$.



Problem 3



Problem 5

3. (106/196) The block on the inclined plane again, this time with springs. A block of mass m sits on an inclined plane of mass M. The block is attached to a spring of spring constant k that is fixed to the plane, and the plane is attached to a spring of the same spring constant k that is attached to a wall. The block and plane slide frictionlessly. Find the normal mode frequencies and vectors of the system. Leave the normal mode vectors unnormalized.

4. (106/196) A planet of mass M, radius R, and uniform density sweeps through a dust cloud of uniform mass density ρ with speed v_0. The dust particles begin at rest with respect to the cloud, so that the only *initial* motion in the problem is the relative motion of the cloud and the planet. The dust cloud's mass density is negligible compared to the planet's. Assume that any dust particle that hits the planet sticks to it. If not for the effect of gravity, the mass accretion rate would just be $\pi R^2 \rho v_0$. But gravity causes dust particles with impact parameter $b > R$ to be attracted to the planet too. Calculate the total cross section for the capture of dust particles onto the planet – essentially, the cross-sectional area of the cloud swept out by the planet – in terms of M, R, G, and v_0, and the corresponding mass accretion rate (in terms of M, R, G, v_0, and ρ). Note that you do *not* need to calculate a differential cross section to do this problem.

5. (106/196) A car of mass m is traveling in the x-direction and maintains constant horizontal speed v. The car goes over a bump whose shape is described by $y = A\,[1 - \cos(\pi x/l)]$ for $0 \le x \le 2\,l$, and $y = 0$ otherwise. (See figure.) The car has shock absorbers that can be treated as a single massless critically damped spring of spring constant k and rest length h. Do the following:

   (a) (5 pts) Calculate an expression for the driving force created by the bump.

   (b) (10 pts) Assume a solution for the vertical motion of the car of the form $y(t) = y_h(t) + y_p(t)$, where $y_h(t)$ is a solution to the corresponding homogeneous equation and $y_p(t)$ is a solution matching the driving force. Write down the form of the solution, write down the equations you would use to determine the coefficients of the terms in the solution, describing how you would use the equations to find the coefficients. Be as explicit and clear as possible – since there won't be a final algebraic answer, the number of points you get will depend on how well we understand what you want to do. Realize that you must determine a solution for both $0 < t < 2l/v$ and $t > 2l/v$. NOTE: do *not* use the Green's function method here, it results in an ugly integral.

   (c) (5 pts) Why is it optimal for the shock absorbers to be critically damped, as opposed to overdamped?

6. (196) A particle of mass m moves in one dimension in a potential energy field $V(q)$ and is retarded by a damping force $F_d(\dot{q}) = -2\,m\,\gamma\,\dot{q}$ proportional to its velocity.

   (a) (5 pts) Show that the equation of motion can also be obtained from the Lagrangian

$$L = e^{2\,\gamma\,t}\left[\frac{1}{2}\,m\,\dot{q}^2 - V(q)\right] \tag{1}$$

   Find the canonical momentum p conjugate to q and show that the Hamiltonian is

$$H = \frac{p^2}{2\,m}\,e^{-2\,\gamma\,t} + V(q)\,e^{2\,\gamma\,t} \tag{2}$$

   (b) (5 pts) For the generating function

$$F_2(q, P, t) = e^{\gamma t}\,q\,P \tag{3}$$

   find the transformed Hamiltonian $K(Q, P, t)$. For an oscillator potential $V(q) = \frac{1}{2}\,m\,\omega^2 q^2$, show that the transformed Hamiltonian yields a constant of the motion

$$K = \frac{P^2}{2\,m} + \frac{1}{2}\,m\,\omega^2 Q^2 + \gamma\,Q\,P \tag{4}$$

   (c) (10 pts) Obtain the solution $q(t)$ for the damped oscillator from the constant of the motion K for the underdamped case. You may need the integral

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \tag{5}$$

7. (196) Suppose that a system with time-independent Hamiltonian $H_0(q,p)$ has imposed on it an external oscillating field, so that the Hamiltonian becomes $H = H_0(q,p) - \epsilon\,q\,\sin\omega t$ where ϵ and ω are constants.

   (a) (5 pts) What is the physical interpretation of $\epsilon \sin \omega t$?

   (b) (5 pts) How are the canonical equations of motion modified?

   (c) (5 pts) Find a canonical transformation that restores the canonical form of the equations of motion – that is, find a generating function and the resulting transformation equations that get rid of the extra time-dependent term in the Hamiltonian. What is the transformed Hamiltonian? Obtain the new equations of motion from the old ones to show that the new equations of motion are in canonical form.

   (d) (5 pts) Also check that the transformation is canonical using Poisson brackets.