Physics 106a/196a – Problem Set 4 – Due Oct 27, 2006

Version 2, October 25, 2006

These problems cover the material in Sections 2.1.9, 2.1.10, and 2.2 of the lecture notes. This includes Lagrangians for nonconservative cases, symmetry transformations and Noether's theorem, and variational calculus and dynamics. Problems 1 and 2 are for 106a students only, 3 and 4 for 106a and 196a students, and 5 and 6 for 196a students only.
version 2: Hint given in Problem 1 about "rightness" of parallelepiped. Corrected sign error in Lagrangian in Problem 3. Noted that hole of radius a in Problem 4 is stationary. Corrected omission of sign on $\vec{\nabla} V$ in Problem 5.

1. (106) Find the dimensions of the parallelepiped of maximum volume circumscribed by

   (a) a sphere of radius R

   (b) an ellipsoid of semiaxes a, b, and c.

   Hint: Is it possible for the parallelepiped to have any non-90° angles? You need not give a rigorous proof.

2. (106) Hand and Finch 1-20. Note that the problem continues on to the next page (parts (d) and (e)). Remember that, if $\theta_{eq} \neq 0$, your Taylor expansion around θ_{eq} is *not* just $\sin(\theta_{eq} + \eta) \approx \eta$ and $\cos(\theta_{eq} + \eta) \approx 1$; go back to the Taylor expansion formula and work it out carefully to make sure you get it correct. In part (e), where it says "Prove that the angular frequency of these small oscillations is ...", you just need to get the equation of motion into the form

   $$\ddot{\eta} + \omega^2 \eta = 0$$

   and identify ω^2. Don't sweat the algebra on getting ω^2 into the right form; getting the SHO equation of motion is the important thing. And notice that you can do parts (d) and (e) starting from Equation 1.101 in Hand and Finch even if you are unable to do some or all of (a), (b), and (c).

3. (106/196) One can obtain a classical derivation of the Schrödinger equation from a variational principle. We will do this in one spatial dimension, and we will also explore how the fact that phase transformation is a symmetry of the Lagrangian leads to conservation of probability. The complication is that, instead of a single independent variable t, we have two independent variables x and t, and instead of a particle path $\vec{r}(t)$, the functions we are going to vary are the wavefunction $\psi(x,t)$ and its conjugate $\psi^*(x,t)$ (or, equivalently, the real part and the imaginary part of ψ).

(a) The appropriate Lagrangian for the problem is

$$\mathcal{L}\left(\psi, \psi^*, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial t}, \frac{\partial \psi^*}{\partial x}, \frac{\partial \psi^*}{\partial t}, x, t\right)$$
$$= \frac{i\hbar}{2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) - \frac{\hbar^2}{2\,m}\left(\frac{\partial \psi}{\partial x}\right)\left(\frac{\partial \psi^*}{\partial x}\right) - V\psi^*\psi \quad (1)$$

. The action is

$$S = \int\int dx\, dt\, \mathcal{L}\left(\psi, \psi^*, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial t}, \frac{\partial \psi^*}{\partial x}, \frac{\partial \psi^*}{\partial t}, x, t\right) \quad (2)$$

where the integral is done over all space and all time. From the above Lagrangian, obtain Schrödinger's equation,

$$-\frac{\hbar^2}{2\,m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\,\psi = i\,\hbar\,\frac{\partial \psi}{\partial t} \quad (3)$$

and its complex conjugate. Hand and Finch Section 2.9 explains how to obtain the Euler-Lagrange equations when there are two independent variables.

(b) Show that the Lagrangian is invariant under the "coordinate" transformation $\psi' = \psi e^{i\,\phi}$ where ϕ is independent of x and t. Figure out how to generalize the derivation of Noether's theorem to show that this invariance leads to the 1-D quantum mechanical continuity equation,

$$\frac{\partial}{\partial t}|\psi|^2 + \frac{\partial j}{\partial x} = 0 \qquad \text{with} \qquad j = \frac{i\,\hbar}{2\,m}\left(\psi\frac{\partial \psi^*}{\partial x} - \psi^*\frac{\partial \psi}{\partial x}\right) \quad (4)$$

NOTE: You must do this via Noether's theorem, not by simply using Schrödinger's equation to evaluate $\frac{\partial}{\partial t}|\psi|^2$.

As you know, $|\psi|^2$ is the quantum mechanical probability density for observing a particle at (x, t) and j is the quantum mechanical probability current. Phase invariance thus gives us conservation of probability. Remarkable!

4. (106/196) A hoop of radius b and mass m rolls without slipping within a stationary circular hole of radius $a > b$ and is subject to gravity. Use as your generalized coordinates the rotation angle ϕ of the hoop and the angular position of the hoop's center θ. Do the following:

(a) Write down the constraint between these two coordinates.

(b) Write down the Lagrangian in terms of these two coordinates. (Be careful here! Be sure you understand all the contributions to the kinetic energy.)

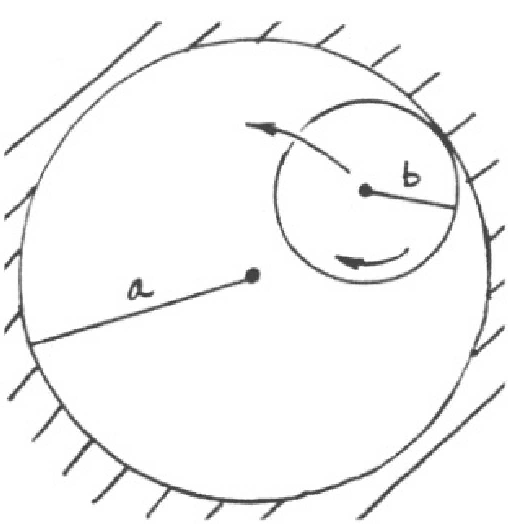
(c) Find the Euler-Lagrange equations with a Lagrange multiplier for the constraint (*i.e.*, don't use the constraint to simplify L, keep L a function of both θ and ϕ).

(d) Find an equation of motion in θ.

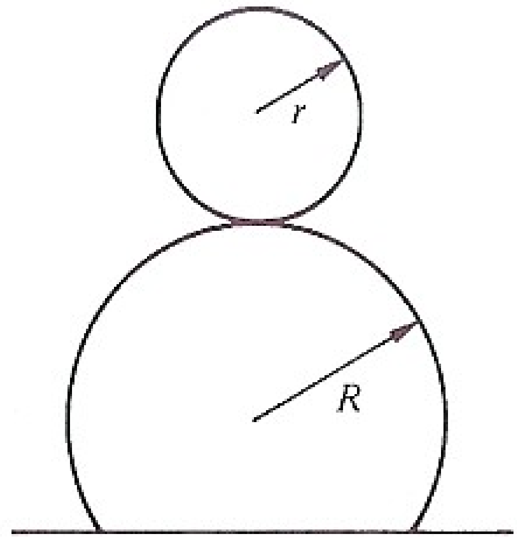
(e) Find the generalized constraint force that makes the hoop roll without slipping.

(f) How would one go about finding the constraint force that keeps the hoop's CM moving on a circular path (don't do it, it's a fair amount of pointless algebra)?

(g) Make the approximation that the motion about the equilibrium point is small and rewrite the equation of motion in simple harmonic oscillator form. What is the frequency of these small oscillations?



Problem 4



Problem 6

5. (196) A rectangular coordinate system with axes x, y, z is rotating with uniform angular velocity ω about the z-axis. A particle of mass m moves under the action of a potential energy $V(x, y, z)$. Do the following:

   (a) Set up the Euler-Lagrange equations of motion.

   (b) Show that these equations can be regarded as the equations of motion of a particle in a fixed coordinate system acted on by the force $-\vec{\nabla}V$ and by a force derivable from a velocity-dependent potential U. You will thereby obtain a velocity-dependent potential for the (fictitious) centrifugal and coriolis forces.

6. (196) A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R as shown in the figure. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder, use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder. You will find it instructive to review Hand and Finch Appendix 1A before starting the problem.