Physics 106b/196b – Problem Set 9 – Due Jan 19, 2007

Version 3: January 18, 2007

This problem set focuses on dynamics in rotating coordinate systems (Section 5.2), with some additional early material on dynamics of rigid bodies (Section 5.3). Again, the length of the problem set is misleading; much of it is in expository material. Problems 1 and 2 are for 106b students only, 3 though 5 for 106b an 196b students, and 6 and 7 are for 196b students only. It is suggested you start thinking about Problem 5 early in the week; it may require a bit of time for you to get your mind around the problem.
Version 2: Now provide expected result for position of CM in Problem 2, and explicitly ask you to calculate it. Even though the problem did not ask for it in v. 1, you need to calculate it to get the right principal moments. Hint added for Problem 4. Problem 5: Additional details given on how to set up coordinate system. Corrected the expected result for kinetic energy (it didn't make sense dimensionally).
Version 3: A couple typos in Problem 5 still – the y components of $\vec{\underline{\omega}}$ and $\vec{\underline{L}}$ were missing a minus sign and there was a 1/2 missing from T. The first one had no impact on the rest of the problem. The second one would make you calculate the incorrect Lagrangian and oscillation frequency in Problem 5b. Also, the explanation of the axis orientation, "At $t = 0$, the xz plane is normal to the $x'y'$-plane with the x axis in the first quadrant of the $x'z'$-plane" is incorrect in an obvious way: the x axis is in the *third* quadrant of the $x'z'$-plane at $t = 0$. Given the late date of these corrections, we will be lenient in grading on any errors that may have resulted from these typos.

1. (106b) Two glorified plug-and-chug problems:

   (a) A wagon wheel with spokes is mounted on a vertical axis so it is free to rotate in the horizontal plane. The wheel is rotating with an angular speed of $\omega = 3.0$ radians/sec. A bug crawls out on one of the spokes of the wheel with a velocity of 0.5 cm/s holding on to the spoke with a coefficient of friction $\mu = 0.30$. How far can the bug crawl out along the spoke before it starts to slip?

   (b) A carousel (counter-clockwise merry-go-round) starts from rest and accelerates at a constant angular accelereration of 0.02 revolutions/s^2. A girl sitting on a bench on the platform 7.0 m from the center is holding a 3.0 kg ball. Calculate the magnitude and direction of the force she must exert to hold the ball 6.0 s after the carousel starts to move. Give the direction with respect to the line from the center of rotation to the girl.

2. (106b) Show that the position of the center-of-mass and the values of the principal moments of inertia of a right circular cone of height h, half-angle a, mass M, and uniform density are

$$\vec{R} = \left(0, 0, \frac{h}{4}\right)$$

$$I_1 = I_2 = \frac{3}{20} M h^2 \left(\frac{1}{4} + \tan^2 \alpha\right) \qquad I_3 = \frac{3}{10} M h^2 \tan^2 \alpha$$

where the center-of-mass is calculated in a coordinate system in which the base of the cone sits in the xy-plane with its center at the origin. Of course, the principal moments must *always* be calculated in a system in which the origin sits at the center of mass, so you have to move origins to then calculate the principal moments.

3. (106b/196b) In Section 5.2.3 of the lecture notes, we derived the Coriolis force for the Foucault's pendulum problem via Lagrangian methods. We assumed ω was constant and we neglected terms of order ω^2, resulting in the loss of the centrifugal and Euler force terms. Let's now generalize this to allow $\vec{\omega}$ to point in an arbitrary direction and to be time-dependent. Write the Lagrangian for this more general case, keeping the ω^2 terms. Find the Euler-Lagrange equations – you should obtain a set of equations equivalent to Equation 5.4 from the lecture notes, which is the equation of motion in the rotating frame with all the fictitious forces included. The canonical momentum you obtain along the way will not be $m\,\underline{\vec{v}}_{body}$; give a physical interpretation of the canonical momentum. Calculate the Hamiltonian (but don't bother to calculate Hamilton's equations). Show that $H = H_{\omega=0} - \underline{\vec{\omega}} \cdot \underline{\vec{l}}_{body}$ where $\underline{\vec{l}}_{body} = \underline{\vec{r}} \times \underline{\vec{p}}_{body}$ and $\underline{\vec{p}}_{body} = \frac{\partial}{\partial \underline{\vec{v}}_{body}} L(\underline{\vec{r}}, \underline{\vec{v}}_{body}, t)$.

   If you have any confusion as to why we use $\_$ and $_{body}$ here in $\underline{\vec{v}}_{body}$ and $\underline{\vec{p}}_{body}$, you should consult someone to clear it up.

   You will find it most convenient to do this problem using index notation for the vector arithmetic. In particular, you will obtain cross products in the kinetic energy like $\underline{\vec{v}}_{body} \cdot (\underline{\vec{\omega}} \times \underline{\vec{r}})$ and $(\underline{\vec{\omega}} \times \underline{\vec{r}}) \cdot (\underline{\vec{\omega}} \times \underline{\vec{r}})$; leave them in cross-product form, and then write them in index notation when you want to take the usual derivatives to get the Euler-Lagrange equations.

4. (106b/196b) Water being diverted during a flood in Helsinki, Finland (latitude 60° N) flows along a diversion channel of width 47 m in the south direction at a speed of 3.4 m/s. On which side is the water highest (from the standpoint of a noninertial system) and by how much? Hint: What relation defines the surface of the water? Consider a mass element of water on the water surface. In what direction must the net force on that element point?

5. (106b/196b) Fun with rolling cones. Apologies for the length of the exposition, hopefully it will prevent you from heading off in a wrong direction. Two parts:

   (a) The right circular cone in Problem 2 above rolls on its side without slipping on a horizontal plane. It completes a circle in time τ. Set up your body and space frames as follows:

   - The space and body frame origins are at the (motionless) apex of the cone; the body frame origin is *not* at the CM of the cone.
   - The body frame z-axis is along the axis of the cone with the $+z$ direction pointing from the base to the apex.
   - The space frame z'-axis is perpendicular to the horizontal plane.
   - At $t = 0$, the cone's line of contact with the horizontal plane coincides with the x'-axis.
   - At $t = 0$, the xz plane is normal to the $x'y'$-plane with the x axis in the third quadrant of the $x'z'$-plane, or, equivalently, the yz plane is inclined at an angle α to the $x'y'$-plane with the y and y' axes coincident at $t = 0$.

Define

$$\omega_p \equiv \frac{2\pi}{\tau} \qquad \Omega \equiv \frac{2\pi}{\tau}\frac{1}{\sin\alpha} \qquad \tilde{I}_1 = I_1 + M\left(\frac{3}{4}h\right)^2 = \frac{3}{20}\,M\,h^2\left(4+\tan^2\alpha\right)$$

Show that the angular velocity, angular momentum, and kinetic energy of the cone in the space and body frames as a function of time are as follows:

$$\underline{\vec{\omega}} = \begin{pmatrix} \omega_p\cos\alpha\cos\Omega t \\ -\omega_p\cos\alpha\sin\Omega t \\ \Omega-\omega_p\sin\alpha \end{pmatrix} \qquad \underline{\vec{\omega}}' = \begin{pmatrix} -\Omega\cos\alpha\cos\omega_p t \\ -\Omega\cos\alpha\sin\omega_p t \\ 0 \end{pmatrix} \qquad \underline{\vec{L}} = \begin{pmatrix} \tilde{I}_1\omega_p\cos\alpha\cos\Omega t \\ -\tilde{I}_1\omega_p\cos\alpha\sin\Omega t \\ I_3\left(\Omega-\omega_p\sin\alpha\right) \end{pmatrix}$$

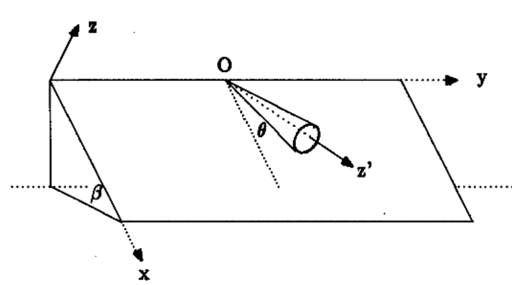
$$\underline{\vec{L}}' = \begin{pmatrix} -\tilde{I}_1\omega_p\cos\alpha\,\sin\alpha\,\cos\omega_p t - I_3\left(\Omega-\omega_p\sin\alpha\right)\cos\alpha\,\cos\omega_p t \\ -\tilde{I}_1\omega_p\cos\alpha\,\sin\alpha\,\sin\omega_p t - I_3\left(\Omega-\omega_p\sin\alpha\right)\cos\alpha\,\sin\omega_p t \\ \tilde{I}_1\omega_p\cos^2\alpha - \tilde{I}_3\left(\Omega-\omega_p\sin\alpha\right)\sin\alpha \end{pmatrix}$$

$$T = T' = \frac{1}{2}\,\omega_p^2\left[\tilde{I}_1\cos^2\alpha + I_3\left(\frac{1}{\sin\alpha}-\sin\alpha\right)^2\right]$$

You may use the results from Problem 2, even if you did not do that problem. You should calculate T and T' in the body and space frame independently to check that they are the same in the two frames. A few key points to keep in mind to avoid confusing yourself:

- $\underline{\vec{\omega}}$, $\underline{\vec{L}}$, and T' are not quantities one can *measure* in the body frame, they are not the angular velocity, angular momentum, and kinetic energy *relative to* the body frame. Rather, the first two are just the angular velocity and angular momentum *relative to* the space frame *written in terms of* the body frame axes, and the kinetic energy in the body frame is just the kinetic energy *calculated from the body frame representations* of $\vec{\omega}$ and $\vec{L}$. All three are just mathematical transformations of their space frame versions.
- We define the body frame's origin to lie at the apex of the cone because, if we used the CM instead, the body frame motion relative to the space frame would include both rotational and noninertial translational motion. The latter is a perfectly reasonable definition, it's just more complicated and not what we are asking for in this problem. The usual reason for going to the body frame is that **I** is diagonal in that frame. Fortunately, in this problem, **I** is also diagonal in the frame we have chosen, so we get the usual body frame simplification – that it is easy to calculate $\vec{L}$ from $\vec{\omega}$ in the body frame – without the complication of the body frame origin undergoing both rotational and noninertial translational motion.

(b) Now, place the cone on a tilted plane as indicated in the figure. Using the results from above, write down the Lagrangian for the problem and find the equations of motion. Find the frequency of small oscillations about the equilibrium point $\theta = 0$.



6. (196b) Gyrocompass. Two parts:

   (a) Let F' be an inertial frame and let F be a frame rotating with respect to F' with constant angular velocity $\vec{\Omega}$. (The two frames' origins are coincident.) If a system of point masses is subject to forces derived from a conservative potential V depending only on the distance to the origin, show that the Lagrangian for the system in terms of the F frame coordinates can be written as

   $$L = T + \underline{\vec{\Omega}} \cdot \underline{\vec{L}} + \frac{1}{2}\,\underline{\vec{\Omega}} \cdot \underline{\mathcal{I}} \cdot \underline{\vec{\Omega}} - V$$

   where T is the kinetic energy and $\vec{L}$ is the angular momentum of the system of mass points *relative to* F. (Do not confuse this with Problem 5, where T and $\vec{L}$ are relative to the inertial system.) $\mathcal{I}$ is of course the inertia tensor of the system of mass points, and $\underline{\mathcal{I}}$ is the coordinate representation of that tensor in F. What is the physical significance of the additional two terms?

   A key point here is that, if we were to take the system of mass points to be a rigid body, F is *not* the body frame of the rigid body. Rather, you can think of F as a frame fixed to the rotating earth (with $\vec{\Omega}$ being the earth's angular velocity), and we are just investigating the motion of a system of point masses in this rotating frame. This part of the problem can be viewed as a generalization of part of Problem 3.

   (b) Suppose that $\vec{\Omega}$ lies in the yz plane in F and that the system of point masses is actually a symmetric top whose symmetry axis is constrained to lie in the xz plane and whose center of mass is fixed to the origin. This arrangement might be realized by, for example, extending through the top's axis a shaft whose ends rest in a frictionless circular track fixed to stay in the xz plane and whose center coincides with the top's CM. The system thus suffers no gravitational torque. Also, take $V = 0$. Show that the top's symmetry axis oscillates about the z axis in a pendulum-like manner and find the frequency of small oscillations.

7. (196b) Consider a charged sphere whose mass m and charge q are both distributed in a spherically symmetric way. That is, the mass and charge densities are each functions of the radius r (but not necessarily the same function). Do the following:

   (a) (3 pts) Show that, if the body rotates in a uniform magnetic field $\vec{B}$, then the torque on it is

   $$\vec{N} = \frac{q\,g}{2\,m\,c}\,\vec{L} \times \vec{B}$$

   in Gaussian units, where g is a numerical constant of order unity (called the gyromagnetic ratio). You will find it useful to take a look at the vector algebraic identities in Section A.3 of the lecture notes. You will also likely run into an expression of the form

   $$\int d^3r\, f(r)\,(\vec{a} \times \vec{r})\left(\vec{r} \cdot \vec{b}\right)$$

   where $\vec{a}$ and $\vec{b}$ do not depend on $\vec{r}$. Realize that this can be written as

   $$\vec{a} \times \left[\int d^3r\, f(r)\,\vec{r}\,\vec{r}^{\,T}\right] \vec{b}$$

   The expression in brackets is similar in form to something we have seen in connection with the inertia tensor.

(b) (1 pts) Show $g = 1$ if the mass density is everywhere proportional to the charge density.

(c) (2 pts) Write an equation of motion for the angular momentum of the body and show that, by introducing a suitable rotating coordinate system, you can eliminate the magnetic torque.

(d) (2 pts) Describe the motion – *i.e.*, how do $\vec{L}$ and $\vec{\omega}$ behave? A new angular velocity should arise – compute its value. Why is it not necessary to take into account this new angular velocity when computing the magnetic torque?

(e) (2 pts) An electron may for some purposes be regarded as a spinning charged sphere of the kind considered in this problem, with $g \approx 2$. Show that, if g were exactly equal to 2, and the electron's spin angular momentum is initially parallel to its linear velocity, then its spin angular momentum would remain parallel to its velocity as it moves through *any* (*i.e.*, possibly nonuniform) magnetic field (assuming the magnetic field is uniform on the size scale of the electron itself). Deviations from $g = 2$ are the result of "virtual particle" effects, such as the electron emitting and reabsorbing a virtual photon (or other, more massive particles such as the carriers of the weak force). A recent experiment measured $g - 2$ of the muon using the behavior you discovered in part (e) and the fact that the decay of the muon to an electron, muon neutrino, and electron antineutrino is correlated with the direction of the muon spin. They found $g - 2$ to be somewhat out of agreement with Standard Model predictions, suggesting the existence of new particles outside the Standard Model.