Physics 125a – Final Exam – Due Dec 14, 2007

Instructions

Material: All lectures through Dec 5, lecture notes Sections 2-9 and 10.1, excluding time-reversal transformations. This corresponds roughly to Shankar Chapters 1, and 4-11, but if material from Shankar was not covered in or pointed to in the lecture notes, it will not be needed here. Review the material ahead of time, consult me, the TAs, your fellow students, or other texts if there is material you are having trouble with.

Logistics: The exam consists of this page plus 2 pages of exam questions, a total of 6 questions. Do not look at the exam until you are ready to start it. Please use a blue book if possible (makes grading easier), but there will be no penalty if you don't have one.

Time: 4 hrs, fixed time. You may take as many breaks as you like, but they may add up to no more than 30 minutes (2 x 15 minutes, 3 x 10 minutes, etc.).

Reference policy: Shankar, official class lecture notes, problem sets and solutions, your own lecture notes or other notes you have taken to help yourself understand the material. No other textbooks, no web searches, no interaction with your fellow students. You may use a computer to write up your exam, but calculators and symbolic manipulation programs are neither needed or allowed. If you write up with a computer, it must be done within the 4-hour exam period; no additional time is allowed for transcription or proofreading. No dispensations will be given for technical difficulties. You may quote without proof any results given in the lecture notes, problem sets, or in Shankar.

Due date: Friday, Dec 14, 4 pm, my office (311 Downs). 4 pm means 4 pm. Late exams will require extenuating circumstances; otherwise, no credit will be given.

Grading: Each problem is 10 points for a total of 60 points. The exam is 1/3 of the class grade.

Suggestions on taking the exam:

- Go through and figure out roughly how to do each problem first; make sure you've got the concept straight before you start writing.
- Don't fixate on a particular problem. They are not all of equal difficulty. Come back to ones you are having difficulty with.
- Don't get buried in algebra (this really should not be an issue on this exam). Get each problem to the point where you think you will get most of the points, then come back and worry about the algebra.

1. Consider a particle subject to a Hamiltonian H whose only two eigenstates are $|\psi_1\rangle$ and $|\psi_2\rangle$ with eigenvalues E_1 and E_2, $E_1 \neq E_2$. Assume they are normalized. Assume that the eigenstates' position-basis representations (wavefunctions) vanish outside the two non-overlapping regions Ω_1 and Ω_2, respectively.

    (a) Show that, if the particle is initially in region Ω_1, then it will stay there forever.

    (b) Show that, if the particle is initially in the state

    $$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left[|\psi_1\rangle + |\psi_2\rangle\right] \tag{1}$$

    then the probability density $|\psi_x(x,t)|^2 = |\langle x|\psi(t)\rangle|^2$ is independent of time.

    (c) Now, assume that the two regions Ω_1 and Ω_2 overlap partially. Starting with the initial state given above, show that the probability density at any point x consists of a constant term plus a term that is a periodic function of time.

2. Consider a one-dimensional system with a real Hamiltonian ($H = H^*$) that occupies a state $|\psi\rangle$ having a wavefunction (position-basis representation) that is real at time $t = 0$ and at a later time $t = t_1$; that is, with $\psi_x(x,t) = \langle x|\psi(t)\rangle$,

    $$\psi_x^*(x,0) = \psi_x(x,0) \qquad \psi_x^*(x,t_1) = \psi_x(x,t_1) \tag{2}$$

    Show that the system is periodic; that is, that there exists a time T for which

    $$\psi_x(x,t+T) = \psi_x(x,t) \tag{3}$$

    Also, show that the energy eigenvalues must be integer multiples of $\frac{2\pi\hbar}{T}$.

3. Consider a one-dimensional potential consisting of an attractive delta function near a wall:

    $$V(x) = \begin{cases} -a\,V_0\,\delta(x-x_0) & x \geq 0 \\ \infty & x < 0 \end{cases} \tag{4}$$

    Let the Hamiltonian be $H = \frac{1}{2m}P^2 + V(X)$. Under what conditions are there bound state solutions, and, if so, how many?

4. For the simple harmonic oscillator, consider the set of states defined by

    $$|z\rangle \equiv e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle \tag{5}$$

    where $|n\rangle$ are the usual SHO eigenstates of energy $E = \hbar\omega\left(n + \frac{1}{2}\right)$ and z is a complex number.

    (a) Show that the $|z\rangle$ states are normalized. Show that they are eigenstates of the annihilation operator a with eigenvalue z.

    (b) Calculate the expectation value of the number operator, $\langle N\rangle$ and the uncertainty $\langle(\Delta N)^2\rangle = \langle(N - \langle N\rangle)^2\rangle$ for a state $|z\rangle$. Show that in the limit $\langle N\rangle \to \infty$, the relative uncertainty $\langle(\Delta N)^2\rangle/\langle N\rangle^2 \to 0$.

(c) Suppose the system is initially in a state $|z\rangle$ at $t = 0$. Calculate the probability of finding the system in this state at a later time $t > 0$. Prove that the evolved state is still an eigenstate of the annihilation operator with a time-dependent eigenvalue. Calculate $\langle N \rangle$ and $\langle N^2 \rangle$ and show that they are time independent.

5. Consider the *dilatation* coordinate transformation, whose generator is the operator

$$D = \frac{1}{2}\left(X\,P + P\,X\right) \tag{6}$$

(Bet you always wondered what this combination was good for!)

(a) Establish the commutation relations

$$[D, X] = -i\,\hbar\,X \qquad [D, P] = i\,\hbar\,P \tag{7}$$

(b) Show that the operators and states transform under this coordinate transformation as

$$X' \equiv e^{-\frac{i}{\hbar}\alpha D}\,X\,e^{\frac{i}{\hbar}\alpha D} = e^{-\alpha}\,X \tag{8}$$

$$P' \equiv e^{-\frac{i}{\hbar}\alpha D}\,P\,e^{\frac{i}{\hbar}\alpha D} = e^{\alpha}\,P \tag{9}$$

$$\psi'_x(x = u) \equiv \langle x = u \,|\, \psi' \rangle \equiv \langle x = u \,|\, e^{-\frac{i}{\hbar}\alpha D} \,|\, \psi \rangle = e^{-\frac{\alpha}{2}}\,\langle x = e^{-\alpha}\,u \,|\, \psi \rangle \equiv e^{-\frac{\alpha}{2}}\,\psi_x(x = e^{-\alpha}\,u) \tag{10}$$

where α is a numerical parameter. You only need to prove the $=$ statements; the $\equiv$ statements are definitions from class. You will need to use the identity

$$e^{-A}\,B\,e^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + +\frac{1}{3!}[[[B, A], A], A] \cdots \tag{11}$$

which you *do not* need to prove.

(c) Is it possible for D to generate symmetry transformations of a one-dimensional Hamiltonian of the form $H = \frac{P^2}{2m} + V(X)$? Why or why not?

6. Consider N noninteracting identical particles. Assume the system's Hamiltonian is the sum of N identical one-particle Hamiltonians with known eigenvalues ϵ_i:

$$H = \sum_{a=1}^{N} H_a \qquad H_a|i\,\rangle_a = \epsilon_i|i\,\rangle_a \tag{12}$$

Assume the $\{\epsilon_i\}$ are ordered in terms of increasing energy; $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_N$.

(a) What is the energy of the ground state of the system if the particles are bosons? And if they are fermions?

(b) Consider the case of three such particles and write down the corresponding ground-state kets.