Physics 125a – Midterm Exam – Due Nov 5, 2007

Instructions

Material: All lectures through Oct 19, Shankar Chapter 1, Lecture Notes Section 3. Review the material ahead of time, consult me, the TAs, your fellow students, or other texts if there is material you are having trouble with.

Logistics: The exam consists of this page plus 2 pages of exam questions, a total of 6 questions. Do not look at the exam until you are ready to start it. Please use a blue book if possible (makes grading easier), but there will be no penalty if you don't have one.

Time: 4 hrs, fixed time. You may take as many breaks as you like, but they may add up to no more than 30 minutes (2 x 15 minutes, 3 x 10 minutes, etc.).

Reference policy: Shankar, official class lecture notes, problem sets and solutions, your own lecture notes or other notes you have taken to help yourself understand the material. No other textbooks, no web searches, no interaction with your fellow students. You may use a computer to write up your exam, but calculators and symbolic manipulation programs are neither needed or allowed. If you write up with a computer, it must be done within the 4-hour exam period; no additional time is allowed for transcription or proofreading. No dispensations will be given for technical difficulties.

Due date: Monday, Nov 5, 4 pm, my office (311 Downs). 4 pm means 4 pm. Late exams will require extenuating circumstances; otherwise, no credit will be given.

Grading: Each problem is 10 points for a total of 60 points. The exam is 1/3 of the class grade.

Suggestions on taking the exam:

- Go through and figure out roughly how to do each problem first; make sure you've got the concept straight before you start writing.
- Don't fixate on a particular problem. They are not all of equal difficulty. Come back to ones you are having difficulty with.
- Don't get buried in algebra (this really should not be an issue on this exam). Get each problem to the point where you think you will get most of the points, then come back and worry about the algebra.

For some of the problems, you will need the *Pauli spin matrices*,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

1. Show that the set of all linear combinations of the Pauli spin matrices with complex coefficients,

$$\gamma = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 \tag{2}$$

where the $\{a_i\}$ can be any complex numbers, form a linear vector space *according to the rules given in the lecture notes*. Remember that, to prove something is a member of the vector space, you just need to show it can be written in the above form. For clarity, use greek letters for the members of the vector space and roman letters for complex coefficients. Do not belabor this; it is indeed not difficult to show all the necessary properties.

2. Define an inner product on the above vector space by

$$\langle v \,| w \,\rangle = \frac{1}{2}\,\mathrm{Tr}\left[v^\dagger w\right] \quad \text{where} \quad \left[v^\dagger w\right]_{ij} = \sum_k \left(v^T\right)^*_{ik} w_{kj} = \sum_k v^*_{ki} w_{kj} \tag{3}$$

That is, transpose and conjugate the matrix v, matrix multiply the result against w, and take the trace of the resulting matrix. Show that this formula satisfies the rules for the definition of an inner product *as given in the lecture notes*. Show that the Pauli spin matrices are an orthonormal basis for the space.

3. Do the following for the Pauli spin matrices, treating them as the matrix representation of operators on 2-d vector space. Note that they are now being treated as *operators*; in the previous two problems, we treated them as elements of a vector space.

   (a) Show they are Hermitian.

   (b) Can the three matrices be simultaneously diagonalized? Why or why not? Your answer should *not* rely on the next part of the problem.

   (c) Find their eigenvalues and eigenvectors and write down separately for each of them the unitary transformation matrix U_i that diagonalizes it.

4. Show

$$e^{i\,\sigma_3\,\theta} = I\cos\theta + i\,\sigma_3\sin\theta \tag{4}$$

where I is the identity operator. Hint: use the matrix representation to calculate σ_3^2.

5. Recall that a *projection operator* P is defined by the requirement $P^2 = P$ and that we have shown that projection operators are Hermitian. Let P be a projection operator. Show the following:

   (a) $I - P$, where I is the identity operator, is also a projection operator.

   (b) Show that the expectation value of P, $\langle P \rangle = \langle \psi \,| P \,| \psi \,\rangle$ where $|\psi\,\rangle$ is arbitrary but normalized ($\langle \psi \,| \psi \,\rangle = 1$), satisfies $0 \le \langle P \rangle \le 1$. (Hint: use that fact that $I - P$ is also a projection operator.)

   (c) Show that the eigenvalues of P can take on only the values 0 or 1.

6. Show $[X^2, K^2] = 2\,(2\,i\,X\,K + 1)$ in two ways:

   (a) By making use of $[X, K] = i\,I$, without applying to any bras or kets.

   (b) In a manner similar to how we proved $[X, K] = i\,I$ in class, using completeness relations and the rules for manipulating delta functions and their derivatives to calculate the matrix element $\langle x\,|\,[X^2, K^2]\,|\,f\,\rangle$, where $|\,f\,\rangle$ is arbitrary and $\langle x\,|$ is a position eigenvector.