Physics 125a – Problem Set 2 – Due Oct 15, 2007

Version 1 – Oct 8, 2007

These problems contain some review on Shankar 1.1-1.7 but focus mainly on Shankar 1.8; Sections 3.1-3.6 of the lecture notes.

1. Let K be the operator defined by $|v\rangle\langle w|$ where $|v\rangle$ and $|w\rangle$ are vectors satisfying $\langle v|w\rangle \neq 0$.

   (a) Under what condition is K Hermitian?

   (b) Calculate K^2. Under what condition is K a projection operator (*i.e.*, $K^2 = K$)?

   (c) Are conditions (a) and (b) above the same?

   (d) Show that K can always be written in the form $K = \lambda P_1 P_2$ where λ is a constant to be calculated and P_1 and P_2 are projection operators.

2. Shankar 1.8.3: Consider the Hermitian matrix

$$\Omega = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

   Show that

   (a) $\omega_1 = \omega_2 = 1$; $\omega_3 = 2$

   (b) $|\omega_3\rangle$ is any vector of the form

$$|\omega_3\rangle = \frac{1}{\sqrt{2a^2}}\begin{bmatrix} 0 \\ a \\ -a \end{bmatrix}$$

   (c) The degenerate ω_1, ω_2 subspace contains all vectors of the form

$$|v\rangle = \frac{1}{\sqrt{b^2 + 2c^2}}\begin{bmatrix} b \\ c \\ c \end{bmatrix}$$

   either by feeding $\omega = 1$ into the equations that yield the eigenvectors or by requiring that the $\omega = 1$ subspace be orthogonal to $|\omega_3\rangle$.

3. Shankar 1.8.6:

   (a) We proved last week that the determinant of a matrix is unchanged by a unitary transformation (change of basis). Show then that

$$\det\,\Omega = \Pi_{i=1}^{n}\omega_i$$

   is the product of the eigenvalues of Ω for a Hermitian or unitary Ω.

(b) Using the analogous invariance of the trace under a unitary transformation, show that, for Hermitian and unitary matrices,

$$\mathrm{Tr}\ \Omega = \sum_{i=1}^{n} \omega_i$$

4. Shankar 1.8.10: By considering the commutator, show that the following Hermitian matrices may be simultaneously diagonalized. Find the eigenvectors common to both and verify that under a unitary transformation to this basis, both matrices are diagonalized.

$$\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Since Ω is degenerate and Λ is not, you must be prudent in deciding which matrix dictates the choice of basis.

5. Repeat Example 1.8.6 from Shankar as outlined in the lecture notes, but allowing for the initial velocities to be nonzero also. Also, in addition to finding $|x(t)\,\rangle$, find $|\dot{x}(t)\,\rangle$.

   Why was the original version of the example, with zero initial velocities, a better analogue to the Schrödinger equation?