Physics 125a – Problem Set 6 – Due Nov 19, 2007

Version 2 – Nov 14, 2007

This problem set provides focuses on the simple harmonic oscillator (Shankar Chapter 7, Lecture Notes Section 6).

Many basic problems in QM can be found in many textbooks – there are only so many solvable basic problems out there. Please refrain from using solutions from other textbooks. Obviously, you will learn more and develop better intuition for QM by solving the problems yourself. We are happy to provide hints to get you through the tricky parts of a problem, but you *must* learn to set up and solve these problems from scratch by yourself.

v. 2: Correct minor typos in Problem 2: used a instead of L as given in drawing, erroneous factor of $1/2$ in H in front of electric potential term.

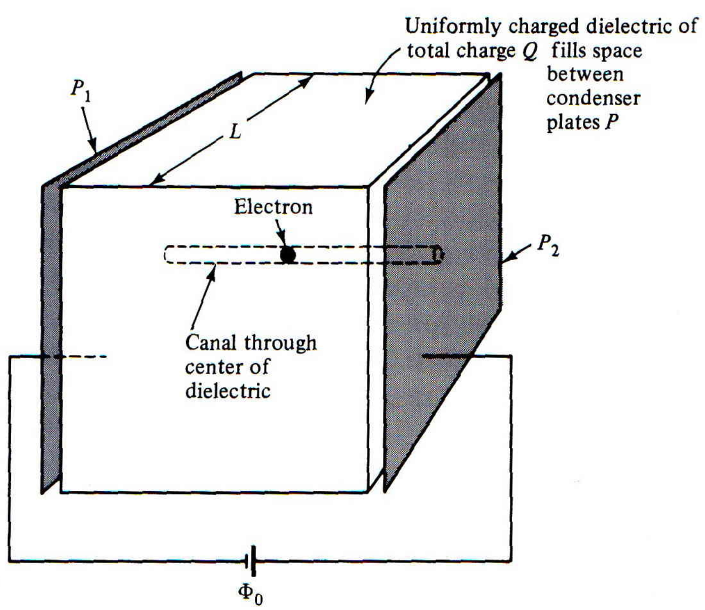
1. For the simple harmonic oscillator, show that the expectation value $\langle X \rangle = \langle \psi \,| X \,| \psi \,\rangle$ for an arbitrary state $|\psi\,\rangle$ can be written in the form

$$\langle X \rangle = A \cos \omega\, t + B \sin \omega\, t \tag{1}$$

   You will need to use the Hermite polynomial recurrence relation

$$H_{n+1}(y) = 2\, y\, H_n(y) - 2\, n\, H_{n-1}(y) \tag{2}$$

2. A large dielectric cube of dielectric constant $\epsilon = 1$ with edge length L is uniformly charged throughout its volume so that it carries a total charge Q. It fills the space between the plates of a capacitor, which have a potential difference Φ_0 across them. An electron is free to move in a small canal drilled in the dielectric normal to the plates (see the figure).



The Hamiltonian for the electron is (with the x origin at the center of the canal and $e = -|e|$)

$$H = \frac{P^2}{2\,m} + \frac{1}{2}\,k\,X^2 + e\,\Phi_0\,\frac{X}{L} \tag{3}$$

Answer the following:

(a) What is the spring constant k in terms of the total charge Q and the cube dimensions L? To evaluate k, use Gauss' Law (neglecting edge effects at the four faces not adjacent to the electrodes).

(b) What are the eigenvalues and eigenvectors of H? Hint: rewrite the potential energy of the electron as

$$V = \frac{1}{2}\,k\,\left(X^2 + 2\,\gamma\,X\right) = \frac{1}{2}\,k\,\left[(X+\gamma)^2 - \gamma^2\right] \qquad \gamma \equiv \frac{e\,\Phi_0}{L\,k} \tag{4}$$

and then change variables to $Z \equiv X + \gamma$. You should ignore the edge of the dielecric in solving this problem; equivalently, assume the classical turning points are well inside the cube so that the wavefunction is not affeted by the edges of the dielectric or the capacitor plates.

3. Show that the eigenstate $|n\,\rangle$ of the SHO satisfies

$$\sqrt{\langle(\Delta X)^2\rangle}\sqrt{\langle(\Delta P)^2\rangle} = \frac{E_n}{\omega} = \hbar\left(n + \frac{1}{2}\right) \tag{5}$$

You may use symmetry arguments whereever appropriate to avoid direct calcultion of expectation values.

4. The SHO with fermions. Assume the standard SHO Hamiltonian,

$$H = \left(a^\dagger a + \frac{1}{2}\right)\hbar\,\omega \tag{6}$$

but, instead of the usual commutation relation for a and $a^\dagger$, assume a and $a^\dagger$ obey the *anticommutation* relation

$$[a, a^\dagger]_+ = a\,a^\dagger + a^\dagger\,a = 1 \tag{7}$$

Note: the above form for H is an *assumption*; you cannot derive these raising and lowering operators and their anticommutation relation from X and P as we did in class. Fermions (and bosons) are fundamentally quantum objects with no classical analogue.

(a) What are the values of $a\,|n\,\rangle$ and $a^\dagger\,|n\,\rangle$ that follow from the above anticommutation relation? (Recall how we found what these states were for the standard SHO.)

(b) Since $\langle H\rangle \geq 0$, for consistency we may again set

$$a\,|0\,\rangle = 0 \tag{8}$$

(Remember, $|0\,\rangle$ is the ground state and 0 is the null vector here.) Combining this fact with your answer to part (a), which are the only nonvanishing states $|n\,\rangle$?

(c) If, in addition to the anticommutation property above, a and $a^\dagger$ obey the relations $[a, a]_+ = [a^\dagger, a^\dagger]_+ = 0$, show that $N^2 = N$ for the number operator $N = a^\dagger a$.

5. The SHO in momentum space

   (a) (Shankar 7.3.7) By setting up an eigenvalue equation for the oscillator in the $\{|p\,\rangle\}$ basis and comparing it to the eigenvalue-eigenvector equation we obtained in the $\{|x\,\rangle\}$ basis, Equation 7.3.2 of Shankar, show that the representation of the eigenstates in the $\{|p\,\rangle\}$ basis may be obtained from the $\{|x\,\rangle\}$-basis representation by the substitution $x \rightarrow p$ and $m\,\omega \rightarrow 1/m\,\omega$. Thus, for example, the $\{|p\,\rangle\}$-basis representation of the ground state is

$$\psi_{0,p}(p) = \left(\frac{1}{m\,\pi\,\hbar\,\omega}\right)^{1/4} e^{-\frac{p^2}{2\,m\,\hbar\,\omega}} \tag{9}$$

   (b) (Shankar 7.5.1) Set up equations to obtain the same $\{|p\,\rangle\}$-basis representations from the states found in the energy-basis formalism (á la Section 7.5 of Shankar and Section 6.4 of the lecture notes).